MODULE 1 STATIC: System of live vectors

UNIT 1 VECTORS

CONTENTS

1.0 INTRODUCTION

A vector could be defined as a quantity which has both magnitude and direction. The vector **a** may be represented geometrically by an arrow of length α drawn from any point in the appropriate direction. In particular, the position of a point *P* with respect to a given origin *O* may be specified by the *position* vector ***r*** drawn from *O* to *P*.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a vector.
- freely discourse some elementary properties of vector.
- know about vector product.
- know about differentiation and integration of vector.
- know about Gradient, Divergence and Curl.
- know about integral theorem.
- know about Curvilinear Co-ordinates.

3.0 MAIN CONTENT

3.1 Definition and Elementary Properties

A vector **a** is a quantity specified by a magnitude, written **a** or $|a|$, and a direction in space. It is to be contrasted with a scalar, which is a quantity specified by a magnitude alone. The vector **a** may be represented geometrically by an arrow of length a drawn from any point in the appropriate direction. In particular, the position of a point *P* with respect to a given origin *O* may be specified by the *position* vector ***r*** drawn from *O* to *P*.

Any vector can be specified, with respect to a given set of Cartesian axes, by three components. If x, y, z are the Cartesian co-ordinates of *P*, then we write r = (x, y, z), and say that x, y, z are the components of **r**. (See Fig. A.I.). We often speak of *P* as 'the point **r**'. When *P* coincides with *O*, we have the *zero vector* **0** = (0, 0, 0) of length 0 and indeterminate director. For a general vector **a**, we write **a** = $[a_{x,} a_y, a_z]$.

The product of a vector **a** and a scalar *c* is c**a** = (ca_x, ca_y, ca_z). If c > 0, it is a vector in the same direction as a, and of length ca; if c < 0, it is the opposite direction, and of length $|c|a$. In particular, if c = 1/a, we have the unit vector in the direction of a written as, à = a /a .

Addition of two vectors **a** and **b** may be defined geometrically by drawing one vector from the head of the other, as in Fig. A. 2. (This is the 'parallelogram law' for addition of forces). Subtraction is defined similarly by Fig. A.3. in terms of components,

$$a + b = (a_x + b_{x,} a_y + b_{y,} a_z + b_z).$$

It is often useful to introduce three unit vectors i, j, k, pointing in the directions of the $x^-, y^-, z-$ axes, respectively. They form what is known as an *orthonomal triad* – a set of three mutually perpendicular vectors of unit length. It is clear from Fig. A.1 that any vector r can be written as a sum of three vectors along the three axes.

$$R = xi + yj + zk. \qquad (1)$$

If θ is the angle between the vectors a and b, then by elementary trigonometry the length of their sum is given by

$$[a + b]^2 = a^2 + b^2 + 2ab \cos \theta.$$

It is useful to define the scalar product **a.b** ('a dot b') as

$$a.b = ab \cos \theta. \qquad (2)$$

Note that this is equal to the length of a multiplied by the projection of **b** on **a**, or vice versa.

In particular, the square of **a** is:

$$a^2 = a.a = a^2. \qquad (3)$$

Thus we can rewrite the relation above as

$$(a+b)^2 = a^2+b^2+2a.b,$$

And similarly

$$(a-b)^2 = a^2+b^2-2a.b.$$

All the ordinary rules of algebra are valid for sums and scalar products of vectors, save one. (For example, the commutative law of addition, a + b = b + a is obvious from Fig. A. 2, and the other laws can be deduced from appropriate figures). The exception is the following: for two scalars, the equation $ab = 0$ implies that either $a = 0 \;\; or \;\; b = 0$ (or, of course, that both = 0), but we can find two non-zero vectors for which a.b = 0. In fact, this is the case if $\theta \frac{1}{2} \pi$, that is if the vectors are orthogonal:

$$a.b = 0 \text{ if } a \perp b.$$

(i.e vector a is perpendicular to vector b)

The scalar products of the unit vectors i, j, k are

$$i^2 = j^2 = k^2 = 1, \qquad (4)$$
$$i.j = j.k = k.i = 0.$$

Thus, taking the scalar product of each in turn with (1), we find

$$i.r = x, \quad j.r = y, \quad k.r = z. \qquad (5)$$

These relations express the fact that the components of r are equal to its projections on the co-ordinate axes.

More generally, if we take the scalar product of two vectors a and b, we find

$$a.b = a_x b_x + a_y b_y + a_z b_z, \qquad (6)$$

and, in particular,

$$r^2 = r^2 = x^2+y^2+z^2. \qquad (7)$$

3.2 The Vector Product

Any two nonparallel vectors **a** and **b** drawn from *0* define a unique axis through 0 perpendicular to the plane containing **a** and **b**. It is useful to define the vector product **a** ^**b** ('**a** cross **b**', sometimes written **a** x **b**) to be a vector along this axis whose magnitude is the area of the parallelogram with edges **a**, **b**,

$$|a \wedge b| = ab \sin \theta \qquad (8)$$

(See Fig. A.4.). To distinguish between the two opposite directions along the axis, we introduce a convention: the direction of $a \wedge b$ is that in which a right-hand screw would move when turned from **a** to **b**.

A vector whose sense is merely conventional, and would be reversed by changing from a right-hand to a left-hand convention is called an axial vector, as opposed to an ordinary or polar vector. For example, velocity and force are polar vectors, but angular velocity is an axial vector (see §5.1). The vector product of two polar vectors is thus an axial vector.

The vector product has one very important, but unfamiliar, property. If we interchange **a** and **b**, we reverse the sign of the vector product,

$$b \wedge a = -a \wedge b. \qquad (9)$$

It is essential to remember this fact when manipulating any expression involving vector products. In particular, the vector product of a vector with itself is the zero vector,

$$a \wedge a = 0.$$

More generally, $a \wedge b$ vanishes if θ = 0 or π ,

$$a \wedge a = 0 \quad if\ a \parallel b.$$

If we choose our co-ordinate axes to be right-handed, then the vector products of i, j, k are

$$\begin{array}{ll} i \wedge i = j \wedge j = k \wedge k = 0, & \\ i \wedge k = k, & j \wedge i = k, \\ j \wedge k = i, & k \wedge j = -i, \\ k \wedge i = j, & i \wedge k = -j. \end{array} \qquad (10)$$

Thus, when we form the vector product of **a** and **b** we obtain

$$a \wedge b = i\,(a_y b_z - a_z b_y) + j(a_z b_x - a_x b_z) + k(a_x b_y - a_y b_x).$$

This relation may conveniently be expressed in the form of a determinant.

$$a \wedge b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \qquad (11)$$

From any three vectors **a, b, c**, we can form the scalar triple product $(a \wedge b).c$. Geometrically, it represents the volume V of the parallele-piped with adjacent edges a, b, c. (See Fig. A.5.) For, if φ is the angle between c and $a \wedge b$, then

$$(a \wedge b).c = |a \wedge b|c \quad \cos \varphi = A\boldsymbol{h} = V,$$

Where A is the area of the base, and h = c cos φ is the height. The volume is reckoned positive if a, b, c form a right-handed triad, and

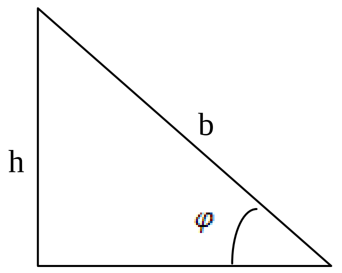

Fig. 5

Negative if they form a left-handed triad. For example, $(i \wedge j.k = 1,$ but $(i \wedge k).j = -1.$

In terms of components, we can evaluate the scalar triple product by taking the scalar product of **c** with (A.11). We find

$$(a \wedge b).c = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \qquad (12)$$

Either from this formula, or from its geometrical interpretation, we see that the scalar triple product is unchanged by any cyclic permutation of **a, b, c,** but changes signs if any pair is interchanged,

$$(a \wedge b).c = (b \wedge c).a = (c \wedge a).b$$

$$= -(b \wedge a).c = -(c \wedge b).a = -(a \wedge c).b. \qquad (13)$$

Moreover, we may interchange the dot and cross,

$$(a \wedge b).c = a;(b \wedge c). \qquad (14)$$

(For this reason, the more symmetrical notation [a, b, c] is sometimes used for the scalar triple product.)

Note that the scalar triple product vanishes if any two vectors are equal, or parallel. More generally, it vanishes if a, b, c are coplanar.

We can also form the vector triple product $(a \wedge b) \wedge c.$ since this vector is perpendicular to $a \wedge b,$ it must lie in the plane of **a** and **b**, and must therefore be a linear combination of these two vectors. It is not hard to show, by writing out the components, that

$$(a \wedge b) \wedge c = b(a.c) - a(b.c). \qquad (15)$$

Similarly,

$$a \wedge (b \wedge c) = b(a.c) - c(a.b). \qquad (16)$$

Note that these expressions are unequal, so that we cannot omit the brackets in a vector triple product. It is useful to notice that in both these formulae the term with positive sign is the middle vector b times the scalar product of the other two.

3.3 Differentiation and Integration of Vectors

We are often concerned with vectors which are functions of some scalar parameter, for example the position of a particle as a function of time, **r**(*t*). The vector distance travelled by the particle in a short time interval Δt is

$$\Delta r = r(t + \Delta t) - r(t).$$

(See Fig. A.6.). The velocity, or derivative with respect to t, is defined just as for scalars, as the limit of a ratio,

$$\acute{r} = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}. \qquad (17)$$

In the limit, the direction of this vector is that of the tangent to the path of the particle, and its magnitude is the speed in the usual sense. In terms of co-ordinates,

$$\dot{r} - (\dot{x}, \dot{y}, \dot{z}).$$

Derivatives of other vectors are defined similarly. In particular, we can differentiate again to form the acceleration vector $\ddot{r}$.

It is easy to show that all the usual rules for differentiating sums and products apply also to vectors. For example,

$$\frac{d}{dt}(a \wedge b) = \frac{da}{dt} \wedge b + a \wedge \frac{db}{dt}$$

Though in this case one must be careful to preserve the order of the two factors, because of the antisymmetry of the vector product.

Note that the derivative of the magnitude of **r**, d**r**/dt, is not the same thing as the magnitude of the derivative $\left|\frac{dr}{dt}\right|$. For example, for a particle moving in a circle, r is constant, so that $\dot{r} = 0,$ but clearly $|\dot{r}|$ is not zero in general. In fact, applying the rule for differentiating a scalar product to $r^2,$ we obtain

$$2r\dot{r} = \frac{d}{dt}\,(r^2) = \frac{d}{dt}\,(r^2) = 2r.\dot{r}$$

Which may also be written

$$\dot{r} = \hat{r}.\dot{r}. \qquad (18)$$

Thus the rate of change of the distance r from the origin is equal to the radial component of the velocity vector.

We can also define the integral of a vector. If **v** = d**r**/dt, then we also write

$$r = \int v \;\; dt,$$

and say that r is the integral of v. If we are given v(t) as a function of time, and the initial value of r, r(t_0), then the position at any later time is given by the definite integral.

$$r(t_0) + \int_{t_0}^{t_1} v_x\,(t)dt. \qquad (19)$$

This is equivalent to the three scalar equations for the components, for example

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} v_x\,(t)dt.$$

One can show, exactly as for scalars, that the integral in (19) may be expressed as the limit of a sum.

3.4 Gradient, Divergence and Curl

There are many quantities in physics which are functions of position in space; for example, temperature, gravitational potential or electric field. Such quantities are known as fields. A scalar field is a scalar function

Ø (x, y, z) of position in space; a vector field is a vector function A (x,y,z). We can also indicate the position in space by the position vector r, and write Ø (**r**) or **A**(**r**).

Now let us consider the three partial derivatives of a scalar field, $\partial\phi|\partial x, \partial\phi|\partial y, \partial\phi|\partial z$. They form the component of a vector field, known as the gradient of Ø and written grad Ø , or $\nabla\phi$ $('del\ \phi')$. To show that they really are the components of a vector, we have to show that it can be defined in a manner which is independent of the choice of axes. We note that if **r** and **r** + d**r** are two neighboring points, then the difference between the values of Ø at these points is

$$d\phi = \phi(r + dr) - \phi(r) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = dr'\nabla\phi. \qquad (20)$$

Now, if the distance $|dr|$ is fixed, then this scalar product takes on its maximum value when d**r** is in the direction of $\nabla\phi$. Hence we conclude that the direction of $\nabla\phi$ is the direction in which Ø increases most rapidly. Moreover, its magnitude is the rate of increase of Ø with distance in this direction. (This is the reason for the name 'gradient'.) Clearly, therefore, we could define $\nabla\phi$ by these properties, which are independent of any choice of axes.

We are often interested in the value of a scalar field Ø evaluated at the position of a particle, Ø(**r**(t)). From (20) it follows that the rate of change of Ø(**r**(t)) is

$$\phi(r(t)) = \dot{r}'\nabla\phi \ . \qquad (21)$$

The symbol ∇ may be regarded as a vector which is also a differential operator (like d/d*x*), given by

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \qquad (22)$$

We can also apply it to a vector field A. The divergence of A is defined to be

$$\text{Div A} = \nabla . A = \frac{\partial A_x}{\partial x} + j\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \qquad (23)$$

And the curl of A to be *

$$curl\ A = \nabla \wedge A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \qquad (24)$$

This latter expression is an abbreviation for the expanded form

$$A = i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$

In particular, we may take A to be the gradient of a scalar field, $A = \nabla\emptyset$. Then its divergence is called the Laplacian of ∅,

$$\nabla.\nabla\emptyset = \nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}. \qquad (25)$$

Just as $a \wedge a = 0$, we find that the curl of a gradient vanishes,

$$\nabla \wedge \nabla\emptyset = 0. \qquad (26)$$

For example, its *z* component is

$$\frac{\partial}{\partial x}\left(\frac{\partial\emptyset}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\emptyset}{\partial x}\right) = 0.$$

Similarly, one can show that the divergence of a curl vanishes,

$$\nabla.(\nabla \wedge A) = 0. \qquad (27)$$

The rule for differentiating products can also be applied to expressions involving ∇ . For example, $\nabla.(A \wedge B)$ is a sum of two terms, in one of which ∇ acts on **A** only, and in the other on **B** only. The gradient of a product of scalar fields can be written

$$\nabla(\emptyset\psi) = \psi\nabla\emptyset + \emptyset\nabla\psi.$$

But, when vector fields are involved, we have to remember that the order of the factors as a product of vectors cannot be changed without affecting the signs. Thus we have

$$\nabla \cdot (A \wedge B) = B.(\nabla \wedge A) - A.(\nabla \wedge B),$$

And similarly

$$\nabla \wedge (\emptyset A) = \emptyset(\nabla \wedge A) - A \wedge (\nabla\emptyset).$$

An important identity, analogous to the expansion of the vector triple product (A.16) is

$$\nabla \wedge (\nabla \wedge A) = \nabla(\nabla.A) - \nabla^2 A, \qquad (28)$$

Where of course

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}.$$

It may easily be proved by inserting the expressions in terms of components.

3.5 Integral Theorems

There are three important theorems for vectors which are generalizations of the fundamental theorem of the calculus,

$$\int_{x_0}^{x_1} \frac{df}{dx} dx = f(x_1) - f(x_0).$$

First, consider a curve *C* in space, running from r_0 to r_1. (see Fig. A.7.) Let the directed element of length along *C* is d**r**. If $\emptyset$ is a scalar field, then, according to (20), the change in $\emptyset$ along this element of length is

$$d\emptyset = dr.\nabla\emptyset.$$

Thus, integrating from r_0 to r_1, we obtain the first of the integral theorems,

$$\int_{r_0}^{r_1} dr.\nabla\emptyset = \emptyset(r_1) - \emptyset(r_0). \qquad (29)$$

The integral on the left is called the line integral of $\nabla\emptyset$ along C. This theorem may be used to relate the potential energy function *V*(**r**) for a conservative force to the work done in going from some fixed point r_0, where V is chosen to vanish, to r. Thus, if $F = -\nabla V$, then

$$V(r) = -\int_{r_0}^{r} dr.F. \qquad (30)$$

When F is conservative, this integral depends only on its end-points, and not on the path *C* chosen between them. Conversely, if this condition is satisfied, we can define V by (30), and the force must be conservative. The condition that two line integrals of the form (30) should be equal whenever their end-points coincide may be restated by saying that the line integral round any closed path should vanish. Physically, this means that no work is done in taking the particle round a loop which returns to its starting point. The integral round a closed loop *C* is usually denoted by the symbol $\oint_c$. Thus we require

$$\oint_c dr.F = 0 \qquad (31)$$

for all closed loops *C*.

This condition may be simplified by using the second of the integral theorems – Stokes' theorem. Consider a curved surface *S*, bounded by the closed curve *C*. If one side of S is chosen to be the 'positive' side, then the positive direction round *C* may be defined by the right-hand screw convention. (See Fig. A.8). Take a small element of the surface, of area d*S*, and let **n** be the unit vector normal to the element, and directed towards its positive side. Then the directed element of area is defined to be d**S** = **n**dS. Stokes' theorem states that if **A** is any vector field, then

$$\iint_S dS.(\nabla \wedge A) = \oint_C dr.A. \qquad (32)$$

The application of this theorem to (31) is immediate. If the line integral round C is required to vanish for all closed curves C, then the surface integral must vanish for all surfaces S. But this is only possible if the integrand vanishes identically. So the condition for a force F to be conservative is

$$\nabla \wedge F = 0. \qquad (33)$$

We shall not prove Stokes' theorem. However, it is easy to verify that it is true for a small rectangular surface. Suppose S is a rectangle in the *xy*-plane of area d*x*d*y*. Then d**S** = **k**d*x*dy, so the surface integral is

$$k.(\nabla \wedge A)\, dxdy = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dxdy. \qquad (A.34)$$

The line integral involves four terms, one from each edge. The two terms arising from the edges parallel to the *x*-axis involve the x component of **A** evaluated for different values of *y*. They therefore contribute

$$A_x(y)dx - A_x(y+dy)dx = -\frac{\partial A_x}{\partial y} dxdy.$$

Similarly, the other pair of edges yield the first term of (34).
We can also find a necessary and sufficient condition for a field **B**(**r**) to have the form of a curl,

$$B = \nabla \wedge A.$$

By (A. 27), such a field must satisfy

$$\nabla . B = 0. \qquad (35)$$

The proof that this is also a sufficient condition (which we shall not give in detail) follows much the same lines as before. One can show it is sufficient that the surface integral of **B** over any closed surface should vanish,

$$\iint_S dS.B = 0,$$

And then use the third of the integral theorems, Gauss' theorem. This states that if *V* is a volume in space bounded by the closed surface *S*, then for any vector field **B**,

$$\iiint_v dV\, \mathbf{\nabla}.B = \iint_s dS.B, \qquad (36)$$

Where dV denotes the volume element dV = dxdydz, and the positive side of S is taken to be the outside.

It is again easy to verify Gauss' theorem for a small rectangular volume dV = dxdydz. The volume integral is

$$\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dxdydz. \qquad (37)$$

The surface integral consists of six terms, one for each face. Consider the faces parallel to the xy-plane, with directed surface elements **k** dxdy and –**k** dxdy. Their contributions involve k.**B** = B_z evaluated for different values of z. thus they contribute

$$B_z(z + dz)dxdy - B_z(z)dxdy = \frac{\partial B_z}{\partial z} dxdydz.$$

Similarly, the other terms of (37) come from the other faces.

3.6 Curvilinear Co-ordinates

One of the uses of the integral theorem is to provide expressions for the gradient, divergence and curl in terms of curvilinear co-ordinates.

Consider a set of orthogonal curvilinear co-ordinates q_1, q_2, q_3, and denotes the elements of length along the three co-ordinate curves by h_1dq_1, h_2dq_2, h_3dq_3. For example, in cylindrical polars,

$$\boldsymbol{h}_p = 1, \qquad \boldsymbol{h}_\varphi = p, \qquad \boldsymbol{h}_z = 1, \qquad (38)$$

and in spherical polars

$$\boldsymbol{h}_r = 1, \qquad \boldsymbol{h}_\theta = r, \qquad \boldsymbol{h}_\varphi = r\sin\theta. \qquad (39)$$

Now consider a scalar field ψ. and two neighbouring points (q_1, q_2, q_3) and (q_1, q_2, q_3 + dq_3). Then the difference between the values of ψ at these points is

$$\frac{\partial \psi}{\partial q_3} dq_3 = d\psi = dr.\mathbf{\nabla}\psi = \boldsymbol{h}_3 dq_3 (\mathbf{\nabla}[\psi)]_3.$$

Where $(\mathbf{\nabla}[\psi)]_3$ is the component of $\mathbf{\nabla}\psi$ in the direction of increasing q_3. Hence we find

$$(\mathbf{\nabla}[\psi)]_3 = \frac{1}{\boldsymbol{h}_3}\frac{\nabla\psi}{\partial q_3}, \qquad (40)$$

with similar expressions for the other components. Thus, in cylindrical and spherical polars, we have

$$\nabla\psi = \left(\frac{\partial\psi}{\partial p}, \frac{1}{p}\frac{\partial\psi}{\partial\varphi}, \frac{\partial\psi}{\partial z}\right). \quad (41)$$

and

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right). \quad (42)$$

To find an expression for the divergence, we use Gauss' theorem, applied to a small volume bounded by the co-ordinate surface. The volume integral is

$$(\nabla . A)h_1 dq_1 h_2 dq_2 h_3 dq_3.$$

In the surface integral, the terms arising from the faces which are surfaces of constant q_3 are of the form $A_3 h_1 dq_1 h_2 dq_2$, evaluated for two different values of q_3. They therefore contribute

$$\frac{\partial}{\partial q_3}(h_1 h_2 h_3) dq_1 dq_2 dq_3.$$

Adding the terms from all three pairs of faces, and comparing with the volume integral, we obtain

$$\nabla . A = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_3 h_1 h_2) + \frac{\partial}{\partial q_3}(h_1 h_2 h_3)\right\}. \quad (43)$$

In particular, in cylindrical and spherical polars,

$$\nabla . A = \frac{1}{p}\frac{\partial(pA_p)}{\partial p} + \frac{1}{p}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}, \quad (44)$$

And

$$\nabla . A = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}. \quad (45)$$

To find the curl, we use Stokes' theorem in a similar way. If we consider a small element of a surface q_3 = constant, bounded by curves of constant q_1 and q_2, then the surface integral is

$$(\nabla \wedge A)_3 h_1 dq_1 h_2 dq_2.$$

In the line integral round the boundary, the two edges of constant q_2 involve $A_1h_1dq_1$ evaluated for different values of q_2, and contribute

$$-\frac{\partial}{\partial q_2}(h_1A_1)dq_1dq_2.$$

Hence, adding the contribution from the other two edges, we obtain

$$(\nabla\wedge A)_3 = \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial q_1}(h_2A_2) - \frac{\partial}{\partial q_2}(h_1A_1)\right\}. \qquad (46)$$

With similar expressions for the other components. Thus, in particular, in cylindrical polars.

$$\nabla\wedge A = \left\{\frac{1}{p}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}, \frac{\partial A_p}{\partial z} - \frac{\partial A_z}{\partial p}, \frac{1}{p}\left(\frac{\partial(pA_\varphi)}{\partial p} - \frac{\partial A_p}{\partial\varphi}\right)\right\}. \qquad (47)$$

And in spherical polars

$$\nabla\wedge A = \left\{\begin{array}{c}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\varphi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right) \\ , \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\end{array}\right\}. \qquad (48)$$

Finally, combining the expressions for the divergence and gradient, we can find the Laplacian of a scalar field. It is

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right\}. \qquad (49)$$

In cylindrical polars

$$\nabla^2\psi = \frac{1}{p}\frac{\partial}{\partial p}\left(p\frac{\partial\psi}{\partial p}\right) + \frac{1}{p^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}. \qquad (50)$$

And, in spherical polars,

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}. \qquad (51)$$

4.0 CONCLUSION

In conclusion, having read through this unit, the student should be able to use vector approach to solve some Engineering, mechanics and physics related problems. More so, students are advised to try all the trial exercises giving to them to enhance their comprehension of the unit. It worth to mention here that; this unit is a perquisite to other units and some other courses in mathematics and physics respectively.

5.0 SUMMARY

What you have learnt in this unit concerns:

that any vector r can be written as a sum of three vectors along the three axes thus: R = $xi + yj + zk.$

If θ is the angle between the vectors a and b, then by elementary trigonometry the length of their sum is given by

$$[a + b]^2 = a^2 + b^2 + 2ab \cos \theta.$$

Obviously, we define the scalar product **a.b** ('a dot b') as

$$a.b = ab \cos \theta.$$

Remark; $(a.b = ab \cos (\theta.$) is equal to the length of a multiplied by the projection of **b** on **a**, or vice versa.

We alsi note that the square of **a** is: $a^2 = a.a = a^2.$

Thus we can rewrite the relation above as $(a+b)^2 = a^2+b^2+2a.b,$

And similarly $(a-b)^2 = a^2+b^2-2a.b.$

All the ordinary rules of algebra are valid for sums and scalar products of vectors, save one. (For example, the commutative law of addition, a + b = b + a is obvious from Fig. A. 2, and the other laws can be deduced from appropriate figures). The exception is the following: for two scalars, the equation $ab = 0$ implies that either $a = 0 \ or \ b = 0$ (or, of course, that both = 0), but we can find two non-zero vectors for which a.b = 0. In fact, this is the case if $\theta \frac{1}{2}\pi$, that is if the vectors are orthogonal:

a.b = 0 if a $\perp$ b.
(i.e vector a is perpendicular to vector b)

The scalar products of the unit vectors i, j, k are

$i^2 = j^2 = k^2 = 1,$
$i.j = j.k = k.i = 0.$

More generally, if we take the scalar product of two vectors a and b, we find

$$a.b = a_x b_x + a_y b_y + a_z b_z,$$

and, in particular,

$$r^2 = r^2 = x^2+y^2+z^2.$$

Vector product

Any two nonparallel vectors **a** and **b** drawn from *0* define a unique axis through 0 perpendicular to the plane containing **a** and **b**.

$$|a \wedge b| = ab \sin \theta$$

The vector product has one very important, but unfamiliar, property. If we interchange **a** and **b**, we reverse the sign of the vector product,

$$b \wedge a = -a \wedge b.$$

Thus, when we form the vector product of **a** and **b** we obtain

$$a \wedge b = i\,(a_y b_z - a_z b_y) + j(a_z b_x - a_x b_z) + k(a_x b_y - a_y b_x).$$

This relation may conveniently be expressed in the form of a determinant.

$$a \wedge b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

From any three vectors **a, b, c**, we can form the scalar triple product $(a \wedge b).c$. Geometrically, it represents the volume V of the parallele-piped with adjacent edges a, b, c. (See Fig. A.5.) For, if φ is the angle between c and $a \wedge b$, then

$$(a \wedge b).c = |a \wedge b|c \cos \varphi = A\boldsymbol{h} = V,$$

Where A is the area of the base, and h = c cos φ is the height.

6.0 TUTOR-MARKED ASSIGNMENT

i. By drawing appropriate figures, prove the following laws of vector (a+b)+c = a+(b+c),
$\lambda(a+b) = \lambda a + \lambda b,$
$a.(b+c) = a.b + a.c.$

Note that a, b, c need not be coplanar.)

ii. Show that $(a \wedge b).(c \wedge d) = a.c\ b.d - a.d\ b.c.$

iii. Evaluate $\nabla \wedge (a \wedge b).$

iv. Prove that $a \wedge (b + c) = a \wedge b + a \wedge c.$ (Hint: Show first that in $a \wedge b, b$ may be replaced by its projection on the plane normal to **a**, and then prove the result for vectors in this plane).

v. Evaluate the components of $\nabla^2 A$ in cylindrical polar co-ordinates using the identity (A. 28). Show that they are not the same as the scalar Laplacians of the components of **A**.

7.0 REFERENCES/FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

Advanced Engineering Mathematics by KREYSZIC.

Generalized function. Mathematical Physics by U. S. Vladinirou.

Vector Analysis and Mathematical Method by S. T. Ajibola. First Published (2006).

Lecture Notes on Analytical Dynamics from LASU (1992).

Lecture Notes on Analytical Dynamics from FUTA (2008).

Lecture Notes on Analytical Dynamics from UNILORIN (1999)

Differential Games by Avner Friedman.

Classical Mechanics by TWB Kibble

UNIT 2 THE ELECTROMAGNETIC FIELD

CONTENTS

1.0 INTRODUCTION

Electromagnetic theory lies outside the scope of this book. However, since we have discussed various examples involving electromagnetic fields, it may be useful to summarize some relevant properties of these fields here. We shall simply quote the results without proof, and we shall not consider the case of dielectric or magnetic media. We shall use Gaussian units, but quote the forms appropriate to SI units in brackets.

2.0 OBJECTIVE

At the end of this unit, you should be able to discussed various examples involving electromagnetic fields.

2.0 MAIN CONTENT

3.1 The Electromagnetic Field

The basic equations of electromagnetic theory are Maxwell's equations. In the absence of dielectric or magnetic media, they may be expressed in terms of two fields, the electric field **E** and the magnetic field **B**. There are two equations involving these fields alone,

$$\nabla \wedge E + \frac{1}{c}\frac{\partial B}{\partial t} = 0, \qquad \left[\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \mathbf{t}} = \mathbf{0}\right] \qquad (1)$$

$$\nabla . B = 0, \qquad [\nabla . B = \mathbf{0}] \qquad (2)$$

And two more involving also the electric charge density p and current density **j**,

$$\nabla \wedge B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} j \qquad \left[\mu_0^{-1} \nabla \wedge B - \varepsilon_0 \frac{\partial E}{\partial t} = j\right] \qquad (3)$$

$$\nabla . E = 4\pi p. \qquad [\varepsilon_0 \nabla . E = p] \qquad (4)$$

The basic set of equations is completed by the Lorentz force equation, which determines the force on a particle of charge q moving with velocity **v**,

$$F = q\left(E + \frac{1}{c} v \wedge B\right). \quad [F = q(E + v \wedge B)] \qquad (5)$$

From (B.2), it follows that there must exist a vector potential A such that

$$B = \nabla \wedge A. \qquad (6)$$

Substituting in (B.1), we then find that there must exist a scalar potential $\emptyset$ such that

$$E = -\nabla\emptyset - \frac{1}{c}\frac{\partial A}{\partial t}. \quad \left[E = -\nabla\emptyset - \frac{\partial A}{\partial t}\right] \qquad (7)$$

These potentials are not unique. If $\wedge$ is any scalar field, then

$$\emptyset' = \emptyset + \frac{1}{c}\frac{\partial \wedge}{\partial t}, \quad \left[\emptyset' = \emptyset + \frac{\partial \wedge}{\partial t}\right]$$

$$A' = A - \nabla \wedge \qquad (8)$$

Define the same fields **E** and **B** as do $\emptyset$ and A. The transformation (B.8) is called a *guage transformation*. In particular, we can always choose $\wedge$ so that the new potentials obey the Lorentz gauge condition

$$\frac{1}{c}\frac{\partial\emptyset'}{\partial t} + \nabla . A' = 0. \quad \left[\frac{1}{c^2}\frac{\partial\emptyset'}{\partial t} + \nabla . A' = 0\right]. \qquad (9)$$

It is only necessary to choose $\wedge$ to be a solution of

$$\frac{1}{c^2}\frac{\partial^2 \wedge}{\partial t^2} - \nabla^2 \wedge = -\left(\frac{1}{c}\frac{\partial\emptyset}{\partial t} + \nabla . A\right). \left[= -\left(\frac{1}{c^2}\frac{\partial\emptyset}{\partial t} + \nabla . A\right)\right]$$

When the Lorentz guage condition is satisfied, we find from (3). (4) and the identity (28) that the potentials satisfy

$$\frac{1}{c^2}\frac{\partial^2\emptyset}{\partial t^2} - \nabla^2\emptyset = 4\pi p, \quad [= \varepsilon_0^{-1} p] \qquad (10)$$

And

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{4\pi}{c} j. \quad [= \mu_0 j] \qquad (11)$$

When there is no electric charge or current density, these are three-dimensional wave equations, which describe a wave propagating with velocity *c*.

For the static case, in which all the fields are time-independent; Maxwell's equations separate into a pair of electrostatic equations,

$$\nabla \wedge E = 0, \qquad \nabla . E = 4\pi p, \qquad [\varepsilon_0^{-1} p] \tag{12}$$

Identical with (6.46) and (6.47), and a pair of magneto static equations,

$$\nabla . B = 0, \qquad \nabla \wedge B = \frac{4\pi}{c} j. \quad [= \mu_0 j] \tag{13}$$

Equation (10) reduces to Poisson's equation (6.48), and (B.11) expresses the vector potential similarly in terms of the current density. The solution of (11) for this case is similar to (6.15), namely

$$A(r) = \frac{1}{c} \iiint \frac{j(r')}{|r - r'|} d^3 r'. \tag{14}$$

[Here and below the SI form is obtained by the replacement $\frac{1}{c} \to \frac{\mu_0}{4\pi}$] Thus, given a static distribution of charges and currents, we can calculate explicitly the scalar and vector potentials, and hence find the fields **E** and **B**.

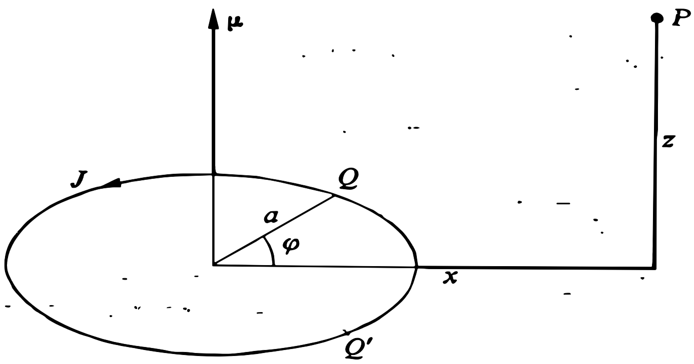

Fig. B.1

As a simple example, we consider a circular current loop of radius *a* in the *xy*-plane, carrying a current **J**. The equation (14) then reduces to a single integration round the loop,

$$A(r) = \frac{J}{c} \oint \frac{dr'}{|r - r'|}. \tag{15}$$

The evaluation of this integral is much simplified by considerations of symmetry. Since the current lies in the xy-plane, A_z is clearly zero. Now let us consider a point P with co-ordinates (x, 0, z). (See fig. B.1) For each point Q on the loop, there will be another point Q', equidistant from P. The contributions of small elements of the loop at Q and Q' to the component A_x will cancel. Thus the only non-zero component at P is A_y. Its value is

$$A_y = \frac{J}{c}\int_0^{2\pi} \frac{a\ \cos\varphi d\varphi}{(r^2 + a^2 - 2ax\ \cos\varphi)^{1/2}}$$

Now we shall assume that the loop is small, so that $a \ll r$. Then the denominator is approximately

$$\left(r^2 - 2ax\ \cos[\varphi)^{-\frac{1}{2}}\right] \approx \frac{1}{r}\left(1 + \frac{ax}{r^2}\cos\varphi\right)$$

Whence

$$A_y = \frac{J}{cr}\int_0^{2\pi}\left(a\ \cos\varphi + \frac{a^2x}{r^2}\ cos^2\varphi\right)d\varphi = \frac{\pi J a^2 x}{cr^3}.$$

It is clear that at an arbitrary point the only non-vanishing component of A will be in the φ direction of polar co-ordinates. If we define the magnetic moment μ of the loop to be

$$\mu = \frac{\pi a^2 J}{c}, \qquad [\mu = \pi a^2 J] \qquad (16)$$

Then the vector potentials is

$$A_r = 0, \qquad A_\theta = 0, \qquad A_\varphi = \frac{\mu \sin\theta}{r^2}. \qquad (17)$$

[Here and below the SI form is obtained by $\mu \to \frac{\mu_0\mu}{4\pi}$.] The co-responding magnetic field is easily evaluated using (A.48).
It is

$$B_r = \frac{2\mu\ \cos\theta}{r^3}, \qquad B_\theta = \frac{\mu \sin\theta}{r^3}, \qquad B_\varphi = 0. \qquad (18)$$

This is a magnetic dipole field. It has precisely the same form as the electric dipole field (6.11).

4.0 CONCLUSION

In conclusion the magnetic dipole field has precisely the same form as the electric dipole field.

5.0 SUMMARY

What you have learned in this unit concerns the basic equations of electromagnetic theory and these are known as Maxwell's equations. As a simple example, we consider a circular current loop of radius *a* in the *xy*-plane, carrying a current **J**.

6.0 TUTOR-MARKED ASSIGNMENT

i. Calculate the vector potential due to a short segment of wire of directed length d**s**, carrying a current *J*, placed at the origin. Evaluate the corresponding magnetic field. Find the force on another segment of length d**s**' carrying current J', at r. Show that this force does not satisfy Newton's third law. (To compute the force, treat the current element as a collection of moving charges).

7.0 REFERENCE/FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

Advanced Engineering Mathematics by KREYSZIC.

Generalized function. Mathematical Physics by U. S. Vladinirou.

Vector Analysis and Mathematical Method by S. T. Ajibola. First Published (2006).

Lecture Notes on Analytical Dynamics from LASU(1992).

Lecture Notes on Analytical Dynamics from FUTA(2008).

Lecture Notes on Analytical Dynamics from UNILORIN(1999)

Differential Games by Avner Friedman.

Classical Mechanics by TWB Kibble

UNIT 3 TENSORS

CONTENTS

1.0 INTRODUCTION

Scalars and vectors are the first two members of a family of quantities known as tensors, and described by 1, 3, 9, 27….. Components. Scalars and vectors are called tensors of rank 0, and of rank 1, respectively. In this appendix, we shall be concerned with the next member of the family, the tensors of rank 2, often called dyadic. We shall use the word tensor in this restricted sense, to mean a tensor of rank 2.

2.0 OBJECTIVES

At the end of this unit, you should be able to recognize scalar and vector as the first two members of Tensors (i.e. to mean a tensor of rank 0 and rank 1 respectively) and the recognition of dyadic to mean a tensor of rank 2

3.0 MAIN CONTENT

3.1 Elementary Properties: The DOT Product

Tensors occur most frequently when one vector **b** is defined as a linear function of another vector **a**, according to

$$\begin{aligned} b_x &= T_{xx}a_x + T_{xy}a_y + T_{xz}a_z, \\ b_y &= T_{yx}a_x + T_{yy}a_y + T_{yz}a_z, \\ b_z &= T_{zx}a_x + T_{zy}a_y + T_{zz}a_z. \end{aligned} \qquad (1)$$

We have already encountered one set of equations of this type-the relations (9.17) between the angular velocity ω and angular momentum **J** of a rigid body.

It will be convenient to introduce a slight change of notation. We write a_1, a_2, a_3 in place of a_x, a_y, a_z, so that (1) may be written

$$b_i = \sum_j T_{ij} a_j, \quad (2)$$

Where i and j run over 1, 2, 3. In this notation, the scalar product of two vectors is

$$a.b = \sum_i a_i b_i. \quad (3)$$

Tensors are commonly denoted by sans-serif capitals, like **T**. The nine components of a tensor **T** may conveniently be exhibited in a square array, or matrix

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \quad (4)$$

Note that the first subscript labels the rows, and the second, the columns.

In view of the similarity between the expressions (2) and (3), it is natural to extend the dot notation, and write (2) in the form

$b = \mathrm{T}.a.$

For instance, the relation (9.17) may be written

$J = 1.\omega$

Where I is the inertia tensor.

We can then form the scalar product of this vector with another vector c, and obtain a scalar

$$c.T.a = \sum_i \sum_j c_i T_{ij} a_j. \quad (5)$$

For example, it follows from (9.22) that the kinetic energy of a rigid body is

$$T = \frac{1}{2}\omega.I.\omega = \frac{1}{2}\sum_i \sum_j \omega_i I_{ij} \omega_j.$$

For any tensor T, we define the transposed tensor $\dot{T}$ by

$\tilde{T}ij = T_{ji}.$

This corresponds to reflecting the array (4) in the leading diagonal. From (5) we see that in general c.T.a is not the same as a.T.c. In fact,

$$a.T.c = \sum_i \sum_j a_i T_{ij} c_j = \sum_j \sum_i c_j \tilde{T}_{ji} a_i.$$

So that

$$a.T.c = c.\breve{T}.a. \qquad (6)$$

Note that he dot always corresponds to a sum over adjacent subscripts.

The tensor T is called symmetric if $\breve{T} = T$, or, equivalently, $T_{ji} = T_{ij}$. In this case, the array (4) is unchanged by reflection in the leading diagonal. An equivalent condition is that, for all vectors a and c,

$$a.T.c = c.T.a. \qquad (7)$$

Similarly, T is called anti-symmetric (or skew-symmetric) if $\breve{T} = -T, or\ T_{ji} = -T_{ij}$. For example, consider the relation giving the velocity of a point in a rotating body,

$$v = \omega \wedge r.$$

This is a linear relation between the components of **r** and **v**, and can therefore be written in the form

$$v = T.r,$$

Where T is some suitable tensor. It is easy to see that its components are given by

$$T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \qquad (8)$$

This tensor is clearly anti symmetric. Note that its diagonal elements T_{ij} are necessarily zero. In fact, any ant symmetric tensor may be associated with an axial vector in this way, and vice versa.

There is a special tensor **1** called the *unit tensor*, or *identity tensor*, which has the property that

$$\mathbf{1}.\boldsymbol{a} = \boldsymbol{a} \qquad (9)$$

For all vectors **a**. Its components are

$$\mathbf{1}_{ij} = \delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j, \end{cases}$$

Or, written out in detail,

$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (10)$$

3.2 Sums and Products; The Tensor Product

The sum of two tensors may be defined in an obvious way. The tensor $R = \propto S + \beta T$ is the tensor with components $R_{ij} = \propto S_{ij} + \beta T_{ij}$. Its effect on a vector **a** is given by

R.a= $\propto$ (**S**.a) + β (**T**.a)

For example, it is easy to show that any tensor **T** can be written as a sum of a sysmmetric tensor **S** and an anti symmetric tensor **A**. in fact, **T** = **S** + **A**, where **S** = $\left(\frac{1}{2}\right)(T + T^{\square})$ and **A** = $\left(\frac{1}{2}\right)(T + T^{\square})$.

We can also define the dot product of two tensors, **S .T**. if c = **S.b** and **b** = **T.a**, then it is natural to write **c** = **S**.(**T.a**) = (**S .T**).a. In terms of components,

$$c_i = \sum_j S_{ij}\left(\sum_k T_{jk}\, a_k\right) = \sum_k\left(\sum_j S_{ij} T_{jk}\right) a_k.$$

Hence **S .T** = **R** is the tensor with components

$$R_{ik} = \sum_j S_{ij}\, T_{jk}. \qquad (11)$$

Once again, the dot signifies summation over adjacent subscripts. Note the rule for constructing the elements of the product: to form the element in the *i*th row and *k*th column of **S .T**, we take the *i*th row of **S**, and the *k*th column of **T**, multiply the corresponding elements, and sum. (This is known as the rule of matrix multiplication.) It is important to realize that, in general, **T .S** ≠ **S .T**. in fact, **T .S** = **Q** has components

$$Q_{ik} = \sum_j T_{ij}\, S_{jk}.$$

There is one special case in which these products are equal, namely the case **S**= **1**. It is easy to see that

1.**T** = **T**.**1** = **T**,

So that **1** plays exactly the same role as the unit in ordinary algebra. From any two vectors **a** and **b** we can form a tensor **T** whose components are $T_{ij} = a_i b_j$. This tensor is written **T** = **a b**, with no dot or cross, and is called the tensor product or dyadic product of **a** and **b**. note that

$\mathbf{T}\cdot\mathbf{c} = (\mathbf{ab})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}),$

So that the brackets are in fact unnecessary. The use of the tensor product allows us to write some earlier results in a different way. For example, for any vector **a**,

$$\mathbf{1}\cdot\mathbf{a} = \mathbf{a} = \mathbf{i}(\mathbf{i}.\mathbf{a}) + \mathbf{j}(\mathbf{j}\cdot\mathbf{a}) + \mathbf{k}(\mathbf{k}\cdot\mathbf{a})$$
$$= (\mathbf{ii} + \mathbf{jj} + \mathbf{kk})\cdot\mathbf{a},$$

So that

$$\mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{1}, \qquad (12)$$

as may easily be verified by writing out the components. Similarly, we may write (9.16) in the form

$$J = \sum m(r^2\omega - rr\cdot\omega) = I\cdot\omega,$$

Where the inertia tensor is given explicitly by

$$\mathbf{I} = \sum m(r^2\mathbf{1} - rr). \qquad (13)$$

It is easy to check that the nine components of this equation reproduce (9.15).

It is clear that if $\mathbf{T}$ = **ab**, then $\tilde{T} = ba$. In particular, it follows that the tensor (13) is symmetric

3.3 Eigenvalues; Diagonalization of a Symmetric Tensor

Throughout this section, we consider a given symmetric tensor $\mathbf{T}$. A vector **a** is called an eigenvector of $\mathbf{T}$ if

$$\mathbf{T}\cdot a = \lambda a, \qquad (14)$$

Where λ is a number called *eigenvalue*. Equivalently, the equation (14) may be written

$$(\mathbf{T} - \lambda 1)\cdot a = 0,$$

Or, written out in full,

$$(T_{11} - \lambda)a_1 + T_{12}a_2 + T_{13}a_3 = 0,$$
$$T_{21}a_1 + (T_{22} - \lambda)a_2 + T_{23}a_3 = 0,$$
$$T_{31}a_1 + T_{32}a_2 + (T_{33} - \lambda)a_3 = 0.$$

These are the same kind of equations that we discussed in Chapter 12 in connection with normal modes. (Compare (12.15). As in that case, the equations are mutually consistent only if the determinant of the coefficients vanishes,

$$\det(T - \lambda 1) = \begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0.$$

When expanded, this determinant is a cubic equation for λ whose three roots are all real, or else one real and two complex conjugates of each other.

We shall now show that the latter possibility can be ruled out. For. Suppose λ is a complex eigenvalue, and a = (a_1, a_2, a_3) the corresponding eigenvalue, whose components may also be complex. We shall denote the complex conjugate eigenvalue by λ*. Then, taking the complex conjugate of

T·a = λa,

We obtain

T·a* = λ*a*,

Where a* = (a_x^*, a_y^*, a_z^*). Multiplying these two equations by **a*** and **a** respectively, we obtain

a*· T·a = λa*·a,

a· T·a* = λ*a·a*.

4.0 CONCLUSION

In this unit we want to conclude by considering a given symmetric tensor **T**. A vector **a which** is called an eigenvector of **T** if

T·a = $\lambda a,$

Where λ is a number called *eigenvalue*. Equivalently, **T**·a = $\lambda a,$ may be written (**T** – λ1)·a = 0,

5.0 SUMMARY

The summary of what you have learnt is as in the conclusion.

6.0 TUTOR-MARKED ASSIGNMENT

i. define the term Tensor
ii. state some properties of Tensor that you are taught
iii. under what consideration can you ascertain that

$\mathbf{T} \cdot a = \lambda a$, may be written as $(\mathbf{T} - \lambda 1) \cdot a = 0$,

7.0 REFERENCES/FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

Advanced Engineering Mathematics by KREYSZIC.

Generalized function. Mathematical Physics by U. S. Vladinirou.

Vector Analysis and Mathematical Method by S. T. Ajibola. First Published (2006).

Lecture Notes on Analytical Dynamics from LASU (1992).

Lecture Notes on Analytical Dynamics from FUTA (2008).

Lecture Notes on Analytical Dynamics from UNILORIN (1999)

Differential Games by Avner Friedman.

Classical Mechanics by TWB Kibble