MODULE 2 DYNAMICS OF SYSTEMS OF PARTICLES

UNIT 1 DISCRETE AND CONTINUOUS SYSTEMS

CONTENTS

1.0 INTRODUCTION

Up to now we have dealt mainly with the motion of an object which could be considered as a particle or point mass. In many practical cases the objects with which we are concerned can more realistically be considered as collections or systems of particles. Such systems are called discrete or continuous according as the particles can be considered as separated from each other or not.

For many practical purposes a discrete system having a very large but finite number of particles can be considered as a continuous system. Conversely a continuous system can be considered as a discrete system consisting of a large but finite number of particles.

2.0 OBJECTIVES

At the end of this unit, you should be able to know about the distinction between Discrete and Continuous Systems with examples.

3.0 MAIN CONTENT

3.1 Density

For continuous systems of particles occupying a region of space it is often convenient to define a mass per unit volume which is called the *volume density* or briefly *density*. Mathematically, if ΔM is the total mass of a volume Δ_T of particles, then the density can be defined as

$$\sigma = \lim_{\Delta_T \to 0} \frac{\Delta M}{\Delta_T} \qquad (1)$$

The density is a function of position and can vary from point to point. When the density is a constant, the systems is said to be of uniform density or simply uniform.

When the continuous system of particles occupy a surface, we can similarly define a surface density or mass per unit area. Similarly when the particles occupy a line [or curve] we can define a mass per unit length or linear density.

3.2 Rigid and Elastic Bodies

In practice, forces applied to systems of particles will change the distances between individual particles. Such systems are often called deformable or elastic bodies. In some cases, however, deformations may be so slight that they may for most practical purposes be considered non-existent. It is thus convenient to define a mathematical model in which the distance between any two specified particles of a system remains the same regardless of applied forces. Such a system is called a rigid body. The mechanics of rigid bodies is considered in Chapters 9 and 10.

3.3 Degrees of Freedom

The number of coordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

a) A particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degrees of freedom is 3.

b) A system consisting of *N* particles moving freely in space requires *3N* coordinates to specify its position. Thus the number of degrees of freedom is *3N*.

c) A rigid body which can move freely in space has 6 degrees of freedom, i.e. 6 coordinates are required to specify the position.

Examples on Degrees of Freedom

1. Determine the number of degrees of freedom in each of the following cases: (a) a particle moving on a given space curve; (b) five particles moving freely in a plane; (c) five particles moving freely in space; (d) two particles connected by a rigid rod moving freely in a plane.

   (a) The curve can be described by the parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$ where s is the parameter. Then the position of a particle on the curve is determined by specifying one coordinate, and hence there is one degree of freedom.

   (b) Each particle requires two coordinates to specify its position in the plane. Thus $5 \cdot 2 = 10$ coordinates are needed so specify the positions of all 5 particles, i.e. the system has 10 degrees of freedom.

   (c) Since each particles requires three coordinates to specify its position, the system has $5 \cdot 3 = 15$ degrees of freedom.

   (d) **Method 1**
   The coordinates of the two particles can be expressed by (x_1, y_1) and (x_2, y_2), i.e. a total of 4 coordinates. However, since the distant between these points is a constant a [the length of the rigid rod], we have $(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$ so that one of the coordinates can be expressed in terms of the others. Thus there are $4 - 1 = 3$ degrees of freedom.

   Method 2
   The motion is completely specified if we give the two coordinates of the centre of mass and the angle made by the rod with some specified direction. Thus there are $2 + 1 = 3$ degrees of freedom

2. Prove that the centre of mass of a system of particles moves as if the total mass and resultant external force were applied at this point.

   Let $\mathbf{F}_v$ be the resultant external force acting on particle v while $f_{v\lambda}$ is the internal force on particle v due to particle λ . We shall assume $\mathbf{f}_{vv} = \mathbf{0}$, i.e. particle v does not exert any force on itself.

   By Newton's second law the total force on particle v is

$$F_v + \sum_{\lambda} f_{v\lambda} = \frac{dp_v}{dt} = \frac{d^2}{dt^2}(m_v r_v) \qquad (1)$$

Where the second term on the left represents the resultant internal force on particle *v* due to all other particles.
Summing over *v* in equation (1), we find

$$\sum_{v} F_v + \sum_{v}\sum_{\lambda} f_{v\lambda} = \frac{d^2}{dt^2}\left\{\sum_{v}(m_v r_v)\right\} \qquad (2)$$

Now according to Newton's third law of action and reaction, $f_{v\lambda} = -f_{\lambda v}$ so that the double summation on the left of (2) is zero. If we then write

$$F = \sum_{v} F_v \quad \text{and} \quad \bar{r} = \frac{1}{M}\sum_{v} m_v r_v \qquad (3)$$

(2) becomes $F = M\dfrac{d^2\bar{r}}{dt^2}$ (4)

Since **F** is the total external force on all particles applied at the centre of mass $\bar{r}$, the required result is proved

3. A system of particles consists of a 3 gram mass located at 91, 0, - 1), a 5 gram mass at (-2, 1, 3) and a 2 gram mass at (3, - 1, 1). Find the coordinates of the centre of mass.

The positive vectors of the particles are given respectively by

$r_1 = i - k$, $r_2 = -2i + j + 3K$, $r_3 = 3i - j + k$

then the centre of mass is given by

$$\bar{r} = \frac{3(i-k) + 5(-2i+j+3k) + 2(3i-j+k)}{3+5+2} = -\frac{1}{10}i + \frac{3}{10}j + \frac{7}{5}k$$

Thus the coordinates of the centre of mass are $\left(-\dfrac{\frac{1}{10,3}}{\frac{10,7}{5}}\right)$.

4. Find the centroid of a solid region $\mathcal{R}$ as in Fig. 7 – 3
Consider the volume element $\Delta\tau_v$ of the solid. The mass of this volume element is

$$\Delta M_v = \sigma_v \Delta\tau_v = \sigma_v \Delta x_v \Delta y_v \Delta z_v$$

Where σ_v is the density [mass per unit volume] and $\Delta x_v, \Delta y_v, \Delta z_v$ are the dimensions of the volume element. Then the centroid is given approximately by

$$\frac{\sum r_v \Delta M_v}{\sum \Delta M_v} = \frac{\sum r_v \sigma_v \Delta \tau_v}{\sum \sigma_v \Delta \tau_v} = \frac{\sum r_v \sigma_v \Delta x_v \Delta y_v \Delta z_v}{\sum \sigma_v \Delta x_v \Delta y_v \Delta z_v}$$

Where the summation is taken over all volume elements of the solid.

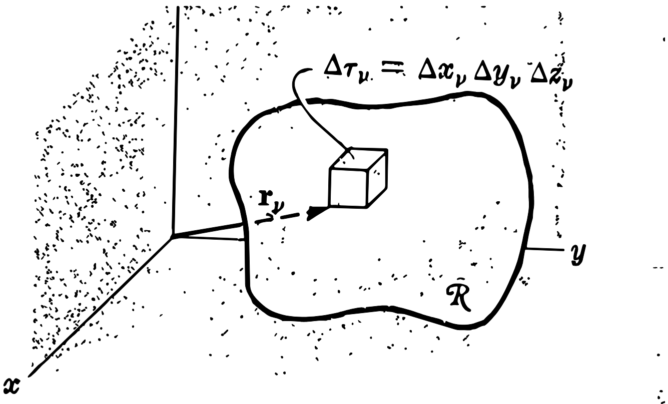

Fig. 7-3

Taking the limit as the number of volume elements becomes infinite in such a way that $\Delta\tau_v \to 0$ or $\Delta x_v \to 0, \Delta y_v \to 0, \Delta z_v \to 0,$ we obtain for the centroid of the solid:

$$\bar{r} = \frac{\int_{\mathcal{R}} r\, dM}{\int_{\mathcal{R}} dM} = \frac{\int_{\mathcal{R}} r\, \sigma d_\tau}{\int_{\mathcal{R}} \sigma\, d_\tau} = \frac{\iiint_{\mathcal{R}} r\, \sigma\, dx\, dy\, dz}{\iiint_{\mathcal{R}} \sigma\, dx\, dy\, dz}$$

Where the integration is to be performed over $\mathcal{R}$, is indicated.

Writing r = x**i** + y**j** + z**k**, $\bar{r} = \bar{x}\boldsymbol{i} + \bar{y}\boldsymbol{j} + \bar{z}\boldsymbol{k},$ this can also be written in component form as

$$\bar{x} = \frac{\int_{\mathcal{R}} r\, dM}{\int_{\mathcal{R}} dM} = \frac{\bar{y}\left(\int_{\mathcal{R}} r\, \sigma d_\tau\right)}{\int_{\mathcal{R}} \sigma\, d_\tau} = \frac{\bar{z}\left(\iiint_{\mathcal{R}} r\, \sigma\, dx\, dy\, dz\right)}{\iiint_{\mathcal{R}} \sigma\, dx\, dy\, dz}$$

5. Find the centre of mass of a uniform solid hemisphere of radius *a*.

z

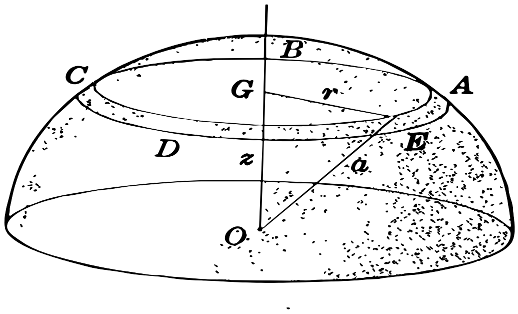

Fig. 7-7

By symmetric the centre of mass lies on the *z* axis [see Fig. 7 – 7]. Subdivided the hemisphere into solid circular plates of radius r, such as *ABCDEA*. If the centre *G* of such a ring is at distance z from the centre *O* of the hemisphere, $r^2 + z^2 = a^2$. Then if *dz* is the thickness of the plate, the volume of each right as

$$\pi r^2 dz = \pi(a^2 - z^2)dz$$

And the mass is $\pi\sigma z(a^2 - z^2)dz$. Thus we have

$$\bar{z} = \frac{\int_{z=0}^{a} \pi\sigma z(a^2 - z^2)}{\int_{z=0}^{a} \pi\sigma(a^2 - z^2)} = \frac{3}{8}a$$

3.4 Centre of Mass

Let r_1, r_2,……., r_N be the position vectors of a system of N particles of masses m_1, m_2, ……., m_N respectively [see Fig. 7 – 1].

The *centre of mass* or *centroid* of the system of particles is define as that point C having position vector

$$\bar{r} = \frac{m_1 r_1 + m_2 r_2 + \cdots .. + m_N r_N}{m_1 + m_2 + \cdots .. + m_N} + \frac{1}{M}\sum_{v=1}^{N} m_v r_v \qquad (2)$$

Where $M = \sum_{v=1}^{N} m_v$ is the total mass of the system. We sometimes use $\sum_v$ or simply $\sum$ in place of $\sum_{v=1}^{N}$.

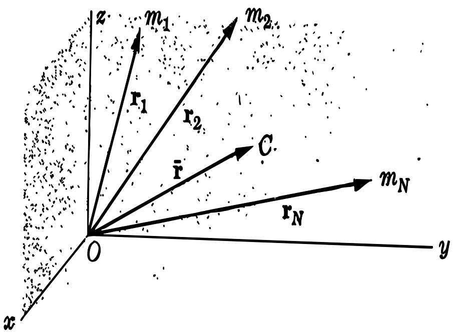

Fig. 7-1

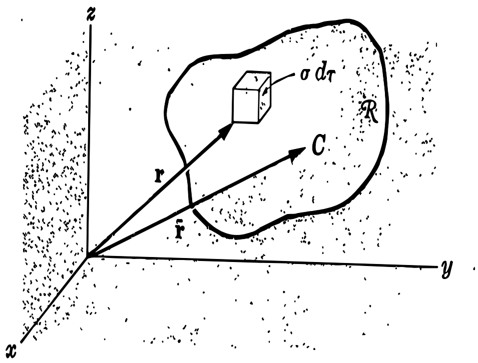

Fig. 7-2

For continuous systems of particles occupying a region $\mathcal{R}$ of space in which the volume density is σ, the centre of mass can be written

$$\check{r} = \frac{\int_{\mathcal{R}} \sigma r d_T}{\int_{\mathcal{R}} \sigma d_T} \qquad (3)$$

Where the integral is taken over the entire region $\mathcal{R}$ [see Fig. 7.2). If we write

$$\bar{r} = \bar{x}i + \bar{y}j + \bar{z}k, \qquad r_v = x_v i + y_v j + z_v k$$

Then (3) can equivalently be written as

$$\bar{x} = \frac{\Sigma m_v x_v}{M}, \qquad \bar{y} = \frac{\Sigma m_v y_v}{M}, \bar{z} = \frac{\Sigma m_v z_v}{M} \qquad (4)$$

And $$\bar{x} = \frac{\int_{\mathcal{R}} \sigma x dr}{M}, \bar{y} = \frac{\int_{\mathcal{R}} \sigma y dr}{M}, \bar{z} = \frac{\int_{\mathcal{R}} \sigma z dr}{M} \qquad (5)$$

where the total mass is given by their

$$M = \sum m_v \qquad (6)$$

or $$M = \int_{\mathcal{R}} \sigma d_T \qquad (7)$$

The integrals in (3), (5) or (7) can be single, double or triple integrals, depending on which may be preferable.

In practice it is fairly simple to go from discrete to continuous systems by merely replacing summations by integrations. Consequently we will present all theorems for discrete systems.

3.5 Centre of Gravity

If a system of particles is in a uniform gravitational field, the centre of mass is sometimes called the centre of gravity.

4.0 CONCLUSION

We shall conclude by saying that In practice it is fairly simple to go from discrete to continuous systems by merely replacing summations by integrations.

5.0 SUMMARY

What you have learnt in this unit concerns: centre of gravity, centre of mass, density, degree of freedom their definitions and examples of each also discussed is their real life applications. Rigid and Elastic Bodies are also taught extensively in this unit.

6.0 TUTOR-MARKED ASSIGNMENT

Show in tabular form the difference between a discrete and a continuous system. Also give examples of each.

7.0 REFERENCES/FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

Advanced Engineering Mathematics by KREYSZIC.

Generalized function. Mathematical Physics by U. S. Vladinirou.

Vector Analysis and Mathematical Method by S. T. Ajibola. First Published (2006).

Lecture Notes on Analytical Dynamics from LASU(1992).

Lecture Notes on Analytical Dynamics from FUTA(2008).

Lecture Notes on Analytical Dynamics from UNILORIN(1999)

Differential Games by Avner Friedman.

Classical Mechanics by TWB Kibble

UNIT 2 MOMENTUM OF A SYSTEM OF PARTICLES

CONTENTS

1.0 INTRODUCTION

When we say system of particles, this refers to centre of mass, the motion of a rotating ax thrown between two jugglers looks rather complicated, and very different from the standard projectile motion alluded to. We deduce from experiment that one point of the ax follows a trajectory described by the standard equations of motion of a projectile. This special point is called the centre of mass of the ax.

2.0 OBJECTIVES

At the end of this unit, you should be able to know momentum of a system of particles as stated in the main contents (3.1-3.8) above.

2.0 MAIN CONTENT

3.1 Momentum of a System of Particles

If $V_v = \frac{dr_v}{dt} = \dot{r}_v$ is the velocity of m_v, the total momentum of the system is defined as

$$P = \sum_{v=1}^{N} m_v V_v = \sum_{v=1}^{N} m_v \dot{r}_v \tag{8}$$

We can show [see Problem 7.3] that

$$P = M\bar{v} = M\frac{d\bar{r}}{dt} = M\dot{\bar{r}} \tag{9}$$

Where $\bar{v} = \frac{d\bar{r}}{dt}$ is the velocity of the centre of mass.

This is expressed in the following

Theorem 1. The total momentum of a system of particles can be found by multiplying the total mass *M* of the system by the velocity $\bar{v}$ of the centre of mass.

3.2 Motion of the Centre of Mass

Suppose that the internal forces between any two particles of the system obey Newton's third law. Then if F is the resultant external force acting on the system, we have

$$F = \frac{dp}{dt} = M\frac{d^2\bar{r}}{dt^2} = M\frac{d\bar{v}}{dt} \qquad (10)$$

This is expressed in

Theorem 2. The centre of mass of a system of particles moves as if the total mass and resultant external force were applied at this point.

3.3 Conservation of Momentum

Putting F = 0 in (10), we find that

$$P = \sum_{v=1}^{N} m_v V_v = \text{constant} \qquad (11)$$

Thus we have

Theorem 3. If the resultant external force acting on a system of particles is zero, then the total momentum remains constant, i.e. is conserved. In such case the centre of mass is either at rest or in motion with constant velocity.

This theorem is often called the principle of conservation of momentum. It is a generalization of Theorem 2 – 8,

3.4 Angular Momentum of a System of particles

The quantity

$$\Omega = \sum_{v=1}^{N} m_v (r_v X V_v) \qquad (12)$$

is called the total angular momentum [of moment of momentum] of the system of particles about origin O.

3.5 The Total External Torque Acting on a System

If F_v is the external force acting on particle v, then r_v X F_v is called the moment of the force F_v or torque about O. the sum

$$\Lambda = \sum_{v=1}^{N} r_v X F_v \qquad (13)$$

is called the *total external torque* about the origin.

3.6 Relation between Angular Momentum and Total External Torque

If we assume that the internal forces between any two particles are always directed along the line joining the particles [i.e. they are central forces], then we can show as in problem 7.12 that

$$\Lambda = \frac{d\Omega}{dt} \qquad (14)$$

Thus we have

Theorem 4. The total external torque on a system of particles is equal to the time rate of change of the angular momentum of the system, provided the internal forces between particles are central forces.

6. Solved examples on Angular Momentum and Torque

Prove theorem 4: The total external torque on a system of particles is equal to the time rate of change of angular momentum of the system, provided that the internal forces between particles are central forces.

we have

$$F_v + \sum_{\lambda} f_{v\lambda} = \frac{dp_v}{dt} = \frac{d}{dt}(m_v v_v) \qquad (1)$$

Multiplying both sides of (1) by $\mathbf{r}_v$ X, we have

$$r_v X F_v + \sum_{\lambda} r_v X f_{v\lambda} = r_v X \frac{d}{dt}(m_v v_v) \qquad (2)$$

Since $$r_v X \frac{d}{dt}(m_v v_v) = \frac{d}{dt}\{m_v(r_v X v_v)\} \qquad (3)$$

(2) becomes $$r_v X F_v + \sum_{\lambda} r_v X f_{v\lambda} = \frac{d}{dt}\{m_v(r_v X v_v)\} \qquad (4)$$

Summing over *v* in (4), we find

$$\sum_v r_v X F_v + \sum_v \sum_\lambda r_v X f_{v\lambda} = \frac{d}{dt}\left\{\sum_v m_v (r_v X v_v)\right\} \qquad (5)$$

Now the double sum in (5) is composed of terms such as

$$r_v \; X \; f_{v\lambda} + r_\lambda \; X \; f_{\lambda v} \qquad (6)$$

Which becomes on writing $f_{\lambda v} = -f_{v\lambda}$ according to Newton's third law,

$$r_v \; X \; f_{v\lambda} - r_\lambda \; X \; f_{v\lambda} = (r_v - r_\lambda) X \; f_{v\lambda} \qquad (7)$$

Then since we suppose that the forces are central, i.e. $f_{v\lambda}$ has the same direction as $r_v - r_\lambda$, it follows that (7) is zero and also that the double sum in (5) is zero. Thus equation (5) becomes

$$\sum_v r_v \; X \; F_v = \frac{d}{dt}\left\{\sum_v m_v (r_v \; X \; v_v)\right\} \quad or \quad \Lambda = \frac{d\Omega}{dt}$$

Where $\Lambda = \sum_v r_v \; X \; F_v, \Omega = \sum_v m_v (r_v \; X \; v_c).$

7. Suppose that the internal forces of a system of particles are conservative and are derived from a potential

$$V_{\lambda v}(r_{\lambda v}) = V_{v\lambda}(r_{v\lambda})$$

Where $f_{\lambda v} = -f_{v\lambda} = \sqrt{(x_\lambda - x_v)^2 + (y_\lambda - y_v)^2 + (z_\lambda - z_v)^2}$ is the distance between particles λ and *v* of the systems.

(a) Prove that $\sum_v \sum_\lambda f_{\lambda v} \cdot dr_v = -\frac{1}{2}\sum_v \sum_\lambda dV_{\lambda v}$ where $f_{\lambda v}$ is the internal force on particles *v* due to particle λ .

(b) Evaluate the double sum $\sum_v \sum_\lambda \int_1^2 f_{\lambda v} \cdot dr_v$ of problem 7.13

(a) The force acting on particle *v* is

$$f_{\lambda v} = -\frac{\partial V_{\lambda v}}{\partial x_v} i - \frac{\partial V_{\lambda v}}{\partial y_v} j - \frac{\partial V_{\lambda v}}{\partial z_v} k = -grad_v \;\; V_{\lambda v} = -\Delta_v \; V_{\lambda v} \qquad (1)$$

The force acting on particle λ is

$$f_{v\lambda} = -\frac{\partial V_{\lambda v}}{\partial x_v} i - \frac{\partial V_{\lambda v}}{\partial y_v} j - \frac{\partial V_{\lambda v}}{\partial z_v} k = -grad_v \ V_{\lambda v} = -\mathbf{\Delta}_v \ V_{\lambda v} = -f_{v\lambda}$$

The work done by these forces in producing the displacements d$\mathbf{r}_v$ and dr_λ of particles v and λ respectively is

$$f_{v\lambda} \cdot dr_v + f_{\lambda v} \cdot dr_\lambda = -\left\{\frac{\partial V_{\lambda v}}{\partial x_v} dx_v + \frac{\partial V_{\lambda v}}{\partial y_v} dy_v + \frac{\partial V_{\lambda v}}{\partial z_v} dz_v + \frac{\partial V_{\lambda v}}{\partial x_\lambda} dx_\lambda + \frac{\partial V_{\lambda v}}{\partial y_\lambda} dy_\lambda + \frac{\partial V_{\lambda_v}}{\partial z_\lambda}\right\}$$

$$= -dV_{\lambda v}$$

Then the total work done by the internal forces is

$$\sum_v \sum_\lambda f_{\lambda v} \cdot dr_v = -\frac{1}{2} \sum_v \sum_\lambda dV_{\lambda v} \qquad (3)$$

The factor $\frac{1}{2}$ on the right being introduced because otherwise the terms in the summation would enter twice.

(b) By integrating (*3*) of part (*a*), we have

$$\sum_v \sum_\lambda \int_1^2 f_{\lambda v} \cdot dr_v = -\frac{1}{2} \sum_v \sum_\lambda \int_1^2 dV_{\lambda v} = V_1^{(int)} - V_2^{(int)} \qquad (4)$$

Where $V_1^{(int)}$ and $V_2^{(int)}$ denote the total internal potentials

$$\frac{1}{2} \sum_v \sum_\lambda V_{\lambda v} \qquad (5)$$

At times t_1 and t_2 respectively.

8. Prove that if both the external and internal forces for a system of particles are conservative, then the principle of conservation of energy is valid.

If the external forces are conservation, then we have

$$F_v = -\Delta V_v \qquad (1)$$

From which $$\sum_v \int_1^2 F_v \cdot dr_v = -\sum_v \int_1^2 dV_v = V_1^{(ext)} - V_2^{(ext)} \qquad (2)$$

Where $V_1^{(ext)}$ and $V_2^{(ext)}$ denote the total external potential

$$\sum_v V_v$$

At times t_1 and t_2 respectively

Using (2) and equation (4) of problem 7.14 (b) in equation (5) of problem 7.13, we find

$$T_2 - T_1 = V_1^{(ext)} - V_2^{(ext)} + V_1^{(int)} - V_2^{(int)} = V_1 - V_2 \tag{3}$$

Where

$$V_1 = V_1^{(ext)} + V_1^{(int)} \text{ and } V_2 = V_2^{(ext)} + V_2^{(int)} \tag{4}$$

Are the respective total potential energies [external and internal] at times t_1 and t_2. We thus find from (3),

$$T_1 + V_1 = T_2 + V_2 \text{ or } T + V = \text{constant} \tag{5}$$

Which is the principle of conservation of energy.

3.7 Conservation of Angular Momentum

Putting $\Lambda = 0$ in (14), we find that

$$\Omega = \sum_{v=1}^{N} m_v (r_v X V_v) = \text{constant} \tag{15}$$

Thus we have

Theorem 5. If the resultant external torque acting on a system of particles is zero, then the total angular momentum remains constant i.e. is conserved

This theorem is often called the principle of conservation of angular momentum. It is the generalization of Theorem earlier discussed.

3.8 Kinetic Energy of a System of Particles

The total kinetic energy of a system of particles is defined as

$$T = \frac{1}{2}\sum_{v=1}^{N} m_v v_v^2 = \frac{1}{2}\sum_{v=1}^{N} m_v \dot{r}_v^2 \tag{16}$$

Work

If $\mathcal{F}_v$ is the force (external and internal) acting on particle v, then the total work done in moving the system of particles from one state [symbolized by 1] to another [symbolized by 2] is

$$W_{12} = \sum_{v-1}^{N} \int_1^2 \mathcal{F}_v \, . \, dr_v \tag{17}$$

As in the case of a single particle, we can prove the following

Theorem 6. The total work done in moving a system of particles from one state where the kinetic energy T_1 to another where the kinetic energy is T_2, is

$$W_{12} = T_2 - T_1 \tag{18}$$

Potential Energy, Conservation of Energy

When all forces, external and internal, are conservative, we can define a total potential energy V of the system. In such case we can prove the following.

Theorem 7: If T and V are respectively the total kinetic energy and total potential energy of a system of particles, then

$$T + V = \text{constant} \tag{19}$$

This is the principle of conservation of energy for systems of particles.

Motion Relative to the Centre of Mass

It is often useful to describe the motion of a system of particles about [or relative to] the centre of mass. The following theorems are of fundamental importance. In all cases primes denote quantities relative to the centre of mass.

Theorem 8: The total linear momentum of a system of particles about the centre of mass is zero. In symbols,

$$\sum_{v=1}^{N} m_v v_v' = \sum_{v=1}^{N} m_v \dot{r}_v' = \mathbf{0} \tag{20}$$

Theorem 9: The total angular momentum of a system of particles about any point O equals the angular momentum of the total mass assumed to be located at the centre of mass plus the angular momentum about the centre of mass. It could be expressed mathematically, thus

$$\Omega = \bar{r}X \;\; M\bar{v} + \sum_{v=1}^{N} m_v \, (r_v' X v_v') \qquad (21)$$

Theorem 10: The total kinetic energy of a system of particles about any point O equals the kinetic energy of translation of the centre of mass [assuming the total mass located there] plus the kinetic energy of motion about the centre of mass. Thus,

$$T \; = \; \frac{1}{2} M\bar{v}^2 + \frac{1}{2}\sum_{v=1}^{N} m_v \, v_v'^2 \qquad (22)$$

Theorem 11: The total external torque about the centre of mass equals the time rate of change in angular momentum about the centre of mass, i.e. equation (14) holds not only for inertial coordinate systems but also for coordinate systems moving with the centre of mass. Consequently,

$$\Lambda' = \frac{d\Omega'}{dt} \qquad (23)$$

If motion is described relative to points other than the centre of mass, the results in the above theorems become more complicated.

Impulse

If **F** is the total external force acting on a system of particles, then

$$\int_{t_1}^{t_2} \mathbf{F} \, dt \qquad (24)$$

is called the *total linear impulse* or briefly *total impulse*. As in the case of one particle, we can prove

Theorem 12: The total linear impulse is equal to the change in linear momentum. Similarly if Λ is the total external torque applied to a system of particles about origin 0, then

$$\int_{t_1}^{t_2} \Lambda \, dt \qquad (25)$$

Is called the total angular impulse. We can then prove

Theorem 13: The total angular impulse is equal to the change in angular momentum.

4.0 CONCLUSION

In conclusion, as in Theorem, 10. The total kinetic energy of a system of particles about any point O equals the kinetic energy of translation of the centre of mass [assuming the total mass located there] plus the kinetic energy of motion about the centre of mass. Thus,

$$T = \frac{1}{2}M\bar{v}^2 + \frac{1}{2}\sum_{v=1}^{N} m_v v_v'^2$$

5.0 SUMMARY

Some thirteen theorems are discussed in this unit thus:

- The total momentum of a system of particles can be found by multiplying the total mass M of the system by the velocity $\bar{v}$ of the centre of mass.
- The centre of mass of a system of particles moves as if the total mass and resultant external force were applied at this point.
- If the resultant external force acting on a system of particles is zero, then the total momentum remains constant, i.e. is conserved. In such case the centre of mass is either at rest or in motion with constant velocity.
- The total external torque on a system of particles is equal to the time rate of change of the angular momentum of the system, provided the internal forces between particles are central forces.
- If the resultant external torque acting on a system of particles is zero, then the total angular momentum remains constant i.e. is conserved
- The total work done in moving a system of particles from one state where the kinetic energy T_1 to another where the kinetic energy is T_2, is
- If T and V are respectively the total kinetic energy and total potential energy of a system of particles, then T+V is a constant.
- The total linear momentum of a system of particles about the centre of mass is zero
- The total angular momentum of a system of particles about any point O equals the angular momentum of the total mass assumed to be located at the centre of mass plus the angular momentum about the centre of mass.
- The total kinetic energy of a system of particles about any point O equals the kinetic energy of translation of the centre of mass [assuming the total mass located there] plus the kinetic energy of motion about the centre of mass.
- The total external torque about the centre of mass equals the time rate of change in angular momentum about the centre of mass.
- The total linear impulse is equal to the change in linear momentum.
- The total angular impulse is equal to the change in angular momentum.

Example

Prove Theorem 10, The total kinetic energy of a system of particles about any point O equals the kinetic energy of the centre of mass [assuming the total mass located there] plus the kinetic energy of motion about the centre of mass.

The kinetic energy relative to O is

$$T = \frac{1}{2}\sum_v m_v v_v^2 = \frac{1}{2}\sum_v m_v (\dot{r}_v' \cdot \dot{r}_v') \qquad (1)$$

but

$$\dot{r}_v = \dot{\bar{r}} + \dot{r}_v' = \bar{V} + v_v'$$

Thus (1) can be written

$$T = \frac{1}{2}\sum_v m_v \{(\bar{v} + v_v') \cdot (\bar{v} + v_v')\}$$

$$= \frac{1}{2}\sum_v m_v \bar{v} \bullet v_v' + \frac{1}{2}\sum_v m_v v_v' \bullet v_v'$$

$$= \frac{1}{2}\left(\sum_v m_v\right)\bar{v}^2 + \bar{v} \cdot \left\{\sum_v m_v v_v'\right\} + \frac{1}{2}\sum_v m_v v_v'^2$$

$$= \frac{1}{2}M\bar{v}^2 + \frac{1}{2}m_v v_v'^2$$

Since $\sum_v m_v v_v' = 0$

6.0 TUTOR-MARKED ASSIGNMENT

i. What can be referred to as being the generalization of Theorems
ii. (2 – 8)?
iii. Prof that the total angular impulse is equal to the change in angular momentum.
iv. state the law of conservation of energy.

7.0 REFERENCE/FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

Advanced Engineering Mathematics by KREYSZIC.

Generalized function. Mathematical Physics by U. S. Vladinirou.

Vector Analysis and Mathematical Method by S. T. Ajibola. First Published (2006).

Lecture Notes on Analytical Dynamics from LASU (1992).

Lecture Notes on Analytical Dynamics from FUTA (2008).

Lecture Notes on Analytical Dynamics from UNILORIN (1999)

Differential Games by Avner Friedman.

Classical Mechanics by TWB Kibble

UNIT 3 Constraints, Holonomic and Non-Holonomic Constraints

CONTENTS

1.0 INTRODUCTION

Often in practice the motion of a particle or system of particles is restricted in some way. For example, in *rigid bodies* [considered in Chapters 9 and 10] the motion must be such that the distance between any two particular particles of the rigid body is always the same.

As another example, the motion of particles may be restricted to curves or surfaces.

The limitations on the motion are often called *constraints*. If the constraint condition can be expressed as an equation

$$\phi(r_1, r_2, \dots\dots, r_N, t) = 0 \qquad (26)$$

connecting the position vectors of the particles and the time, then the constrain is called *holonomic*. If it cannot be so expressed it is called *non-holonomic*.

2.0 OBJECTIVES

At the end of this unit, you should be able to discussed the following:

- virtual Displacements
- statics of a System of particles. Principle of Virtual Work
- equilibrium in Conservative Fields. Stability of Equilibrium
- D'Alembert's Principle

3.0 MAIN CONTENT

3.1 Virtual Displacements

Consider two possible configurations of a system of particles at a particular instant which are consistent with the forces and constraints. To go from one configuration to the other, we need only give the $_v$th particle a displacement δr_v from the old to the new position. We call δr_v a *virtual displacement* to distinguish it from a *true displacement* [denoted by d$\mathbf{r}_v$] which occurs in a time interval where forces and constraints could be changing. The symbol δ has the usual properties of the differential d; for example, $\delta(\sin\theta) = \cos\theta\ \delta\theta$.

3.2 Statics of a System of particles. Principle of Virtual Work

In order for a system of particles to be in equilibrium, the resultant force acting on each particle must be zero, i.e. $\mathbf{F}_v$ = 0. It thus follows that $F_v \cdot \delta r_v$ = 0 where $F_v \cdot \delta r_v$ is called the virtual work. By adding these we then have

$$\sum_{v=1}^{N} F_v \cdot \delta r_v = \mathbf{0} \qquad (27)$$

If constraints are present, then we can write

$$F_v = F_v^{(a)} + F_v^{(c)} \qquad (28)$$

Where $F_v^{(a)}$ and $F_v^{(c)}$ are respectively the *actual force* and *constraint force* acting on the $_v$th particle. By assuming that the virtual work of the constraint forces is zero [which is true for rigid bodies and for motion on curves and surfaces without friction], we arrive at

Theorem 14. A system of particles is in equilibrium if and only if the total virtual work of the actual forces is zero, i.e. if

$$\sum_{v=1}^{N} F_v^{(a)} \cdot \delta r_v = \mathbf{0} \qquad (29)$$

This is often called the *principle of virtual work*.

3.3 Equilibrium in Conservative Fields. Stability of Equilibrium

The results for equilibrium of a particle in a conservative force field can be generalized to systems of particles. The following theorems summarize the basic results.

Theorem 15. If V is the total potential of a system of particles depending on coordinates $q_1, q_2, \ldots\ldots.,$ then the system will be in equilibrium if

$$\frac{\partial V}{\partial q_1} = 0, \frac{\partial V}{\partial q_2} = 0, \ldots \ldots . \qquad (31)$$

Since the virtual work done on the system is

$$\delta V = \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \cdots \ldots.$$

(31) is equivalent to the principle of virtual work.

Theorem 16. A system of particles will be in stable equilibrium if the potential *V* is a minimum.

In case *V* depends on only one coordinate, say q_1, sufficient are

$$\frac{\partial V}{\partial q_1} = 0, \qquad \frac{\partial^2 V}{\partial q_1^2} > 0$$

Other cases of equilibrium where the potential is not a minimum are called unstable.

3.4 D'Alembert's Principle

Although Theorem 14 as stated applies to the statics of a system of particles, it can be restated so as to give an analogous theorem for dynamics. To do this we note that according to Newton's second law of motion,

$$F_v = \dot{P}_v \quad or \quad F_v - \dot{P}_v = 0 \qquad (30)$$

Where p_v is the momentum of the $_v$th particle. The second equation amounts to saying that a moving system of particles can be considered to be in equilibrium under a force $F_v - \dot{P}_v$. i.e. the actual force together with the added force $-\dot{P}_v$ which is often called the reversed effective force on particle *v*. By using the principle of virtual work we can then arrive at

Theorem 17. A system of particles moves in such a way that the total virtual work

$$\sum_{v=1}^{N} (F_v^{(a)} - \dot{P}_v) \cdot \delta r_v = 0 \qquad (32)$$

With this theorem, which is often called D'Alembert's principle, we can consider dynamics as a special case of statics.

Example

Motion Relative to the Centre of Mass

(1) Let r_v' and v_v' be respectively the position vector and velocity of particle v relative to the centre of mass. Prove that (a) $\sum_v m_v r_v' = 0,$ (b) $\sum_v m_v v_v' = 0.$

(a) Let $\mathbf{r}_v$ be the position vector of particle v relative to 0 and $\bar{r}$ the position vector of the centre of mass C relative to O. Then from the definition of the centre of mass,

$$\bar{r} = \frac{1}{M}\sum_v m_v r_v \qquad (1)$$

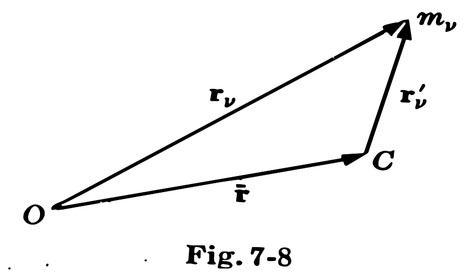

Fig. 7-8

Where $M = \sum_v m_v$. From Fig. 7 – 8 we have

$$r_v = r_v' + \bar{r} \qquad (2)$$

Then substituting (2) into (1), we find

$$\bar{r} = \frac{1}{M}\sum_v m_v (r_v' + \bar{r}) = \frac{1}{M}\sum_v m_v r_v' + \bar{r}$$

From which $\sum_v m_v r_v' = \mathbf{0}$ (3)

(b) Differentiating both sides of (3) with respect to t, we have $\sum_v m_v v_v' = 0.$

Example 2

In each of the following cases whether the constraint is holonomic or non-holonomic and give a reason for your answer: (a) a bead moving on a circular wire; (b) a particle sliding down an inclined plane under the influence of gravity; (c) a particle sliding down a sphere from a point near the top under the influence of gravity.

(a) The constraint is holonomic since the bead, which can be considered a particle, is constrained to move on the circular wire.

(b) The constraint is holonomic since the particle is constrained to move along a surface which is in this case a plane

(c) the constraint way of seeing this is to note that r is the position vector of the particle relative to the centre of the sphere as origin and a is the radius of the sphere, then the particles moves so that $r^2 \geqq a^2$. This is a non-holonomic constraint since it is not of the form (26), page 170. An example of a holonomic constraint would be $r^2 = a^2$.

4.0 CONCLUSION

In conclusion, In order for a system of particles to be in equilibrium, the resultant force acting on each particle must be zero, i.e. $\mathbf{F}_v = 0$

5.0 SUMMARY

The summaries of what you have learnt are as contained in theorems 14 – 17 above thus:

Theorem 14. A system of particles is in equilibrium if and only if the total virtual work of the actual forces is zero, Called principle of virtual work.

Theorem 15. If V is the total potential of a system of particles depending on coordinates q_1, q_2, ……., then the system will be in equilibrium if

$\frac{\partial V}{\partial q_1} = 0, \frac{\partial V}{\partial q_2} = 0, \ldots \ldots$ which is equally equivalent to virtual work.

Theorem 16. A system of particles will be in stable equilibrium if the potential *V* is a minimum. and

Theorem 17. A system of particles moves in such a way that the total virtual work given as

$$\sum_{v=1}^{N} (F_v^{(a)} - \dot{P}_v) \cdot \delta r_v = \mathbf{0}$$

which is often called D'Alembert's principle

6.0 TUTOR-MARKED ASSIGNMENT

i. Explain the term virtual displacement
ii. Define D'Alembert's principle
iii. define centre of mass
iv. what do you understand by the momentum of system of particle
v. Explain the terms holonomic and nonholonomic constraints.

7.0 REFERENCES/FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

Advanced Engineering Mathematics by KREYSZIC.

Generalized function. Mathematical Physics by U. S. Vladinirou.

Vector Analysis and Mathematical Method by S. T. Ajibola. First Published (2006).

Lecture Notes on Analytical Dynamics from LASU (1992).

Lecture Notes on Analytical Dynamics from FUTA (2008).

Lecture Notes on Analytical Dynamics from UNILORIN (1999)

Differential Games by Avner Friedman.

Classical Mechanics by TWB Kibble