MODULE 3 THE SIMPLE PENDULUM

UNIT 1 THE SIMPLE PENDULUM

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1.0 INTRODUCTION

The simple pendulum is one of the most common examples of *simple harmonic motion*, at least as far as laboratory observation of oscillatory motions is concerned. A *harmonic motion* is one for which the restoring force obeys Hooke's law, provided the displacement from equilibrium position is small. In that case the displacement, velocity and acceleration towards the equilibrium position are represented by simple sinusoidal functions of time or linear combinations of them. The term *simple* comes into the definition as a result of the fact that the amplitude and therefore energy of the system is conserved (constant) when dissipative (friction type) forces are negligible. Then the curves of the dynamic variables such as displacement, velocity and acceleration will be pure sine or cosine curves.

We are interested in this type of motion because, as you will recall from your college physics, vibratory motion is one of the four fundamental motions in nature. Vibratory or periodic motion is a prototype of the motions of most physical systems. The structures of buildings, bridges and crystals such quartz used for the construction of your wrist watch are in a state of vibration at all times. The motion of electrons in an antenna that transmits or receives a radio signal is vibratory.

In this unit you will study the simple mathematical formulation of this important type of motion and discuss the properties of the solutions of its differential equation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- derive the equation of a simple pendulum

- show that the equation of a simple pendulum is a particular case of the more general equation of a simple harmonic oscillator
- demonstrate an understanding of the dependence of the period of a simple pendulum on the length and local gravitational acceleration.
- Solve simple problems involving the simple solutions of the equation of the simple pendulum.
- discuss elastic systems in terms of Hooke’s law
- calculate the energy stored in an elastic system
- show that a spring force is conservative.

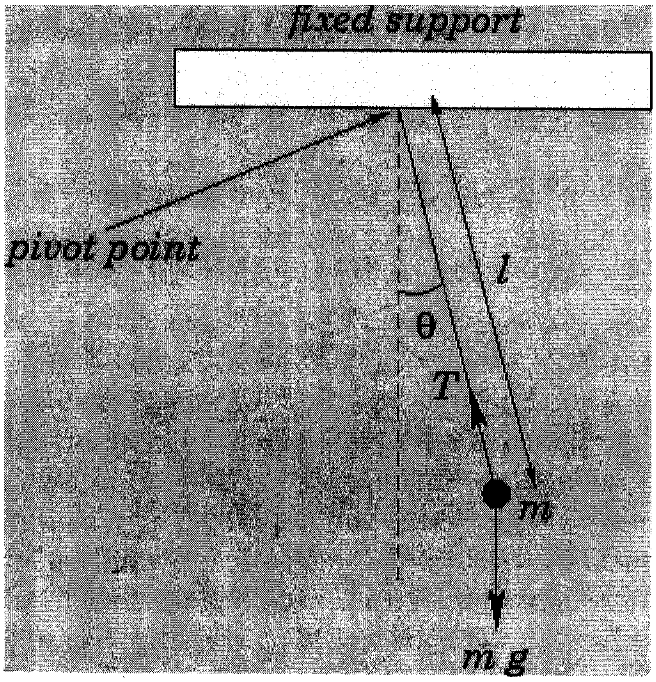
3.0 MAIN CONTENT

3.1 The Simple Pendulum

Consider a mass m suspended from a light inextensible string of length l, such that the mass is free to swing from side to side in a vertical plane, as shown in Fig. a. This setup is known as a *simple pendulum*. Let θ be the angle subtended between the string and the downward vertical. Obviously, the equilibrium state of the simple pendulum corresponds to the situation in which the mass is stationary and hanging vertically down (i.e., θ = 0). The angular equation of motion of the pendulum is simply

$$I \ddot{\theta} = \tau \qquad (523)$$

where I is the moment of inertia of the mass, and τ is the torque acting on the system. For the case in hand, given that the mass is essentially a point particle, and is situated a distance *l* from the axis of rotation (i.e., the pivot point), it is easily seen that $I = ml^2$.



The two forces acting on the mass are the downward gravitational force, *mg*, and the tension, *T*, in the string. Note, however, that the tension makes no contribution to the torque, since its line of action clearly passes through the pivot point. From simple trigonometry, the line of action of the gravitational force passes a distance $l \sin\theta$ from the pivot point. Hence, the magnitude of the gravitational torque is $m\, g\, l \sin\theta$. Moreover, the gravitational torque is a *restoring torque:* i.e., if the mass is displaced slightly from its equilibrium state (i.e., $\theta = 0$) then the gravitational force clearly acts to push the mass back toward that state. Thus, we can write

$$\tau = -m\, g\, l \sin\theta. \qquad 524$$

Combining the previous two equations, we obtain the following angular equation of motion of the pendulum:

$$l\, \ddot{\theta} = -g \sin. \qquad (525)$$

Unfortunately, this is *not* the simple harmonic equation. Indeed, the above equation possesses no closed solution which can be expressed in terms of simple functions.

Suppose that we restrict our attention to relatively *small* deviations from the equilibrium state. In other words, suppose that the angle θ is constrained to take fairly small values. We know, from trigonometry, that for $|\theta|$ less than about 6^0 it is a good approximation to write

$$\sin\theta \simeq \theta. \qquad (526)$$

Hence, in the *small angle limit,* reduces to

$$l\, \ddot{\theta} = -g\, \theta, \qquad (527)$$

which is in the familiar form of a simple harmonic equation. Comparing with our original simple harmonic equation, and its solution, we conclude that the angular frequency of small amplitude oscillations of a simple pendulum is given by

$$\omega = \sqrt{\frac{g}{l}}\,. \qquad (528)$$

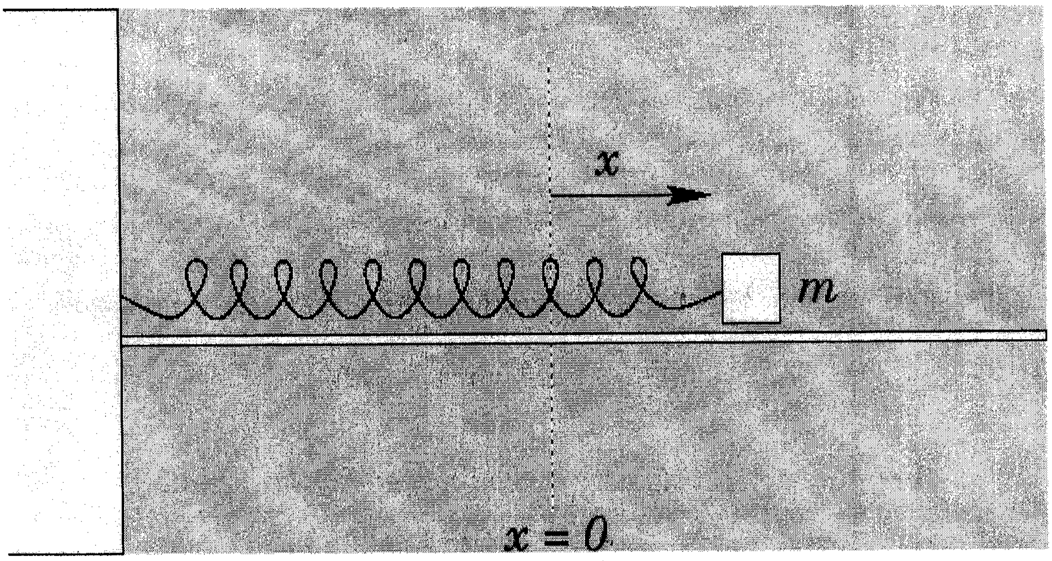
In this case, the pendulum frequency is dependent only on the length of the pendulum and the local gravitational acceleration, and is independent of the mass of the pendulum and the amplitude of the pendulum swings (provided that $\sin\theta \simeq \theta$ remains a good approximation). Historically, the simple pendulum was the basis of virtually all accurate time-keeping devices before the advent of electronic clocks. Simple pendulums can also be used to measure local variations in g.

3.2 Hooke's Law

Consider a mass *m* which slides over a horizontal frictionless surface. Suppose that the mass is attached to a light horizontal spring whose other end is anchored to an immovable object. See Fig. Let x be the extension of the spring: i.e., the difference between the spring's actual length and its unstretched length. Obviously, x can also be used as a coordinate to determine the horizontal displacement of the mass. According to Hooke's law, the force *f* that the spring exerts on the mass is directly proportional to its extension, and always acts to reduce this extension. Hence, we can write

$$f = - kx, \qquad (159)$$

where the positive quantity *k* is called the *force constant* and measures the *stiffness* of the spring. Note that the minus sign in the above equation ensures that the force always acts to reduce the spring's extension: e.g., if the extension is positive then the force acts to the left, so as to shorten the spring.



Mass on a spring

According to Eq. (140), the work performed by the spring force on the mass as it moves from displacement x_A to x_B is

$$W = \int_{x_A}^{x_B} f(x)dx = -k \int_{x_A}^{x_B} xdx = -\left[\frac{1}{2}kx_B^2 - \frac{1}{2}kx_A^2\right].$$

Note that the right-hand side of the above expression consists of the difference between two factors: the first only depends on the final state of the mass, whereas the second only depends on its initial state. This is a sure sign that it is possible to associate a *potential energy* with the spring force. Equation (155), which is the basic definition of potential energy, yields

$$U(x_B) - U(x_A) = - \int_{x_A}^{x_B} f(x)dx = \frac{1}{2}kx_B^2 - \frac{1}{2}kx_A^2. \quad (161)$$

Hence, the potential energy of the mass takes the form

$$U(x) = \frac{1}{2}kx^2. \quad (162)$$

Note that the above potential energy actually represents energy stored by the spring – in the form of mechanical stresses – when it is either stretched or compressed. Incidentally, this energy must be stored *without loss*, otherwise the concept of potential energy would be meaningless. It follows that the spring force is another example of a *conservative force*.

It is reasonable to suppose that the form of the spring potential energy is somehow related to the form of the spring force. Let us now explicitly investigate this relationship. If we let $x_B \to x$ and $x_A \to 0$ then Eq. (161) gives

$$U(x) = -\int_0^x f(x')dx' \quad (163)$$

We can differentiate this expression to obtain

$$f = -\frac{dU}{dx}. \quad (164)$$

Thus, in 1-dimension, a conservative force is equal to minus the derivative (with respect to displacement) of its associated potential energy. This is a quite general result. For the case of a spring force: $U = (1/2)\, kx^2$, so $f = -dU/dx = -kx$.

As is easily demonstrated, the 3-dimensional equivalent to Eq. (164) is

$$F = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right). \quad (165)$$

For example, we have seen that the gravitational potential energy of a mass *m* moving above the Earth's surface is $U = m\, g\, z$, where *z* measures height off the ground. It follows that the associated gravitational force is

$$f = (0, 0, -mg). \quad (166)$$

In other words, the force is of magnitude $m\, g$, and is directed vertically downward.

The total energy of the mass shown in Fig. 42 is the sum of its kinetic and potential energies:

$$E = K + U = K + \frac{1}{2}kx^2. \qquad (167)$$

Of course, *E* remains constant during the mass's motion. Hence, the above expression can be rearranged to give

$$K = E - \frac{1}{2}kx^2. \qquad (168)$$

Since it is impossible for a kinetic energy to be negative, the above expression suggests that $|x|$ can never exceed the value

$$x_0 = \sqrt{\frac{2E}{k}}. \qquad (169)$$

Here, x_0 is termed the *amplitude* of the mass's motion. Note that when x attains its maximum value x_0, or its minimum value $-x_0$, the kinetic energy is momentarily zero (i.e., K = 0).

2.0 CONCLUSION

As in the summary.

5.0 SUMMARY

In this unit you have been introduced to the equation of a simple pendulum. It is a particular case of the more general equation of a simple harmonic oscillator.

The period of the motion is independent of its mass but depends on its length and the value of the local gravitational acceleration.

Simple harmonic oscillations are observed only for small displacements from their equilibrium positions. Their restoring forces will then obey Hook's law of elasticity.

For an elastic system, the work done by the elastic forces manifests as the change in its potential energy. The total energy is the sum the kinetic and potential energies interchanged as the elastic system is alternately stretched and compressed.

6.0 TUTOR-MARKED ASSIGNMENT

i. Explain the term **Hooke's Law**
ii. show that a spring force is conservative.

7.0 REFERENCES / FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

Advanced Engineering Mathematics by KREYSZIC.

Generalized function. Mathematical Physics by U. S. Vladinirou.

Vector Analysis and Mathematical Method by S. O. Ajibola. First Published (2006).

Lecture Notes on Analytical Dynamics from LASU (1992).

Lecture Notes on Analytical Dynamics from FUTA (2008).

Lecture Notes on Analytical Dynamics from UNILORIN (1999)

Differential Games by Avner Friedman.

Classical Mechanics by TWB Kibble