MODULE 4

UNIT 1 MOTION ALONG A CURVE

CONTENTS

1.0 INTRODUCTION

In this unit, we shall study the motion of an object, or particle, moving in space. The object may be a car speeding around a racetrack, an electron being propelled through a linear accelerator, or a satellite in orbit. We assume that the motion takes place in a fixed coordinate system and that the object can be located by specifying a single point, its centre of gravity

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand the Motion along a curve
- Understand and solve simple problems in Position, Velocity, and Acceleration of system of particles along curves.

3.0 MAIN CONTENT

3.1 Motion along Curves

In this unit, we study the motion of an object, or particle, moving in space. The object may be a car speeding around a racetrack, an electron being propelled through a linear accelerator, or a satellite in orbit. We assume that the motion takes place in a fixed coordinate system and that the object can be located by specifying a single point, its centre of gravity.

3.1.1 Position, Velocity, and Acceleration

The three basic notions for analyzing motion are position, velocity, and acceleration. As a particle moves along a path, we suppose that the coordinates (x,y,z) of its position are twice differentiable functions of time

$x = x(t) \qquad y = y(t) \qquad z = z(t)$

The vector function

$r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

from the origin to the particle is called the **position function** of the particle. Figure 14-36 shows the position of a particle at time *t* and at another time $t + \Delta t$.

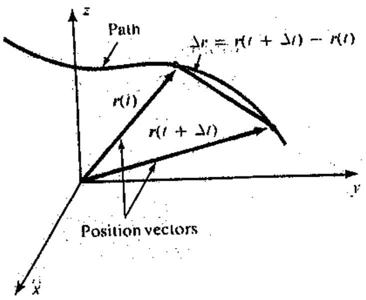

Figure 14-36: Δr is the change in position from time *t* to $t + \Delta t$.

The displacement vector $\Delta r = r(t + \Delta t) - r(t)$ represents the *change in position.* The scalar multiple $\Delta r/\Delta t$ represents the *average* change in position from time *t* to $t + \Delta t$, and the average change in position is called the *average velocity* over the time period Δt. Now, just as in the case of motion along a line, we define the *(instantaneous)* **velocity** to be the limit of the average velocities as Δt approaches 0; that is

$$\text{velocity} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$$

According to (14.36) in the previous section, the limit on the right is the *vector* r'(t). If v(t) denotes the velocity at time *t*, it follows that

$V(t) = r'(t)$

Thus, velocity is the rate of change of position with respect to time. Furthermore, the rate of change of velocity with respect to time is called the **acceleration** and is denoted by **a;** that is,

$a(t) = v'(t) = r''(t)$

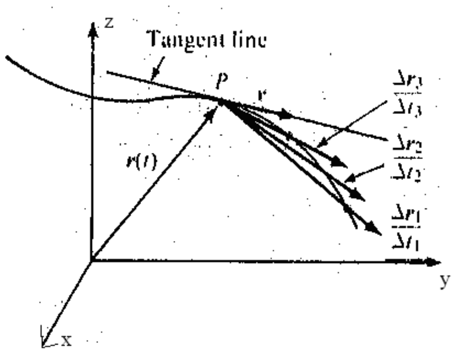

Figure 14-37: The vectors $\Delta r/\Delta t \to v$ as $\Delta t \to 0$; v is a direction vector for the line tangent to the path at *P*.

The discussion above is completely consistent with our earlier discussion of position, velocity, and acceleration. Notice, however, that all three are now considered to be vectors and can be represented by arrows. The position vector always has its tail at the origin, but the velocity and acceleration vectors are considered to have their tails at the location of the particle. Moreover, *the arrow representing the velocity is always tangent to the path.* To see why this is so, we suppose that the particle is at point *P* at time *t*. Figure 14-37 indicates that as $\Delta t \to 0$, the vectors $\Delta r/\Delta t$ approach a direction vector of the tangent line through *P*. It follows that this direction vector is the velocity vector **v.** Figure 14-38 shows some typical velocity and acceleration vectors at various points on the path. Notice that acceleration vectors usually point toward the concave side of the path.

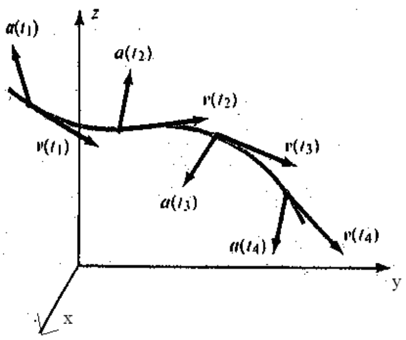

Figure 14-38: Typical velocity and acceleration vectors on the path of motion.

The **speed** v of a particle is defined to be the rate of change of distance (along the path) with respect to time. Speed has magnitude only and is, therefore, a scalar. If the particle starts at time t_0, then the distance *s* it travels along the path from t_0 to time *t* is given by the arc length formula (14.32).

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$$

It follows from the Fundamental Theorem that v(t) = s′(t) = $\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$

But the expression on the right is the length of r′(t) = v(t) and, therefore,

speed = v(t) = $|v(t)|$

The entire discussion above can be summarized as follows:

For a particle travelling through space, we have

(1) r(t) is the position vector; its tail is at the origin and its tip traces out the path.
(2) v(t) = r′(t) is the velocity vector; it is tangent to the path.
(3) a(t) = v′(t) = r″(t) is the acceleration vector; it usually points toward the concave side of the path.
(4) v(t) = s′(t) = $|v(t)|$ is the speed.

Example 1

The position vector of a particle is r(t) = cos *t***i** + sin *t***j** + *t***k**.
Find its velocity, speed, and acceleration at any time *t*.

Solution:

The path of the particle is a circular helix.

v(t) = r′(t) = -sin *t***i** + cos *t***j** + **k**
speed = $|v'(t)|$ = $\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$ = $\sqrt{2}$
a(t) = v′(t) = -cos *t***i** – sin *t***j**

Note: The acceleration is not **0** even though the speed is constant. The reason is that the velocity is constantly changing direction. Also notice that v(t) · a(t) = 0, which means that **v** and **a** are always orthogonal .see the diagram below.

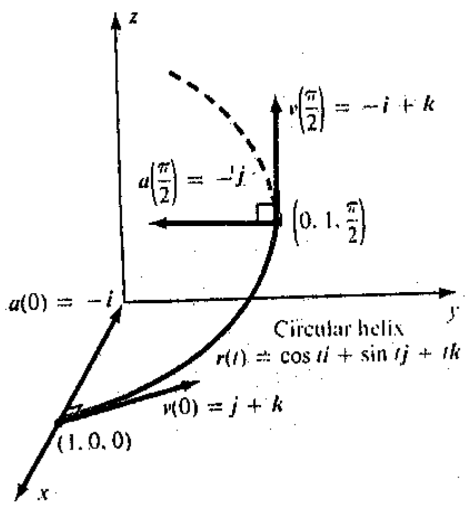

Figure 5

4.0 CONCLUSION

In conclusion, the motion of an object, moving in space. It was assumed that the motion takes place in a fixed coordinate system and that the object can be located by specifying a single point, its centre of gravity.

5.0 SUMMARY

Summarily, we have studied the motion of an object, or particle, moving in space. The object may be a car speeding around a racetrack, an electron being propelled through a linear accelerator, or a satellite in orbit.

6.0 TMA

i. Define the following **Position, Velocity, and Acceleration of a satellite in orbit.**
ii. **Give a brief introduction of motion in a circle.**
iii. The position vector of a particle is $r(t) = \tan t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$. Find its velocity, speed, and acceleration at time $t=30s$.

7.0 REFERENCES / FURTHER READING

Theoretical Mechanics by Murray, R. Spiegel.

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UNIT 2 CIRCULAR MOTIONS WITH CONSTANT SPEED

CONTENTS

1.0 INTRODUCTION

A Circular Motion with Constant Speed is a phenomena in which a particle moves with constant (uniform) speed (velocity) in a circular path

2.0 OBJECTIVES

At the end of this unit, you should be able to define and solve examples on Circular Motion with Constant Speed. Here the acceleration vector always points toward the Centre of the circle (*centripetal) acceleration and/or* the acceleration vector always points toward the circumference of the circle (centrifugal) acceleration.

3.0 MAIN CONTENT

3.1 Circular Motion with Constant Speed

If a particle moves with constant speed in a circular path, then the acceleration vector always points toward the centre of the circle; this is called *centripetal acceleration*. To see why this is so, we observe that a circular path lies in a single plane, and we might as well consider the circle to be in the xy-plane with radius **r** and centre at the origin (Figure 14-40). Since the speed is constant, the angle θ

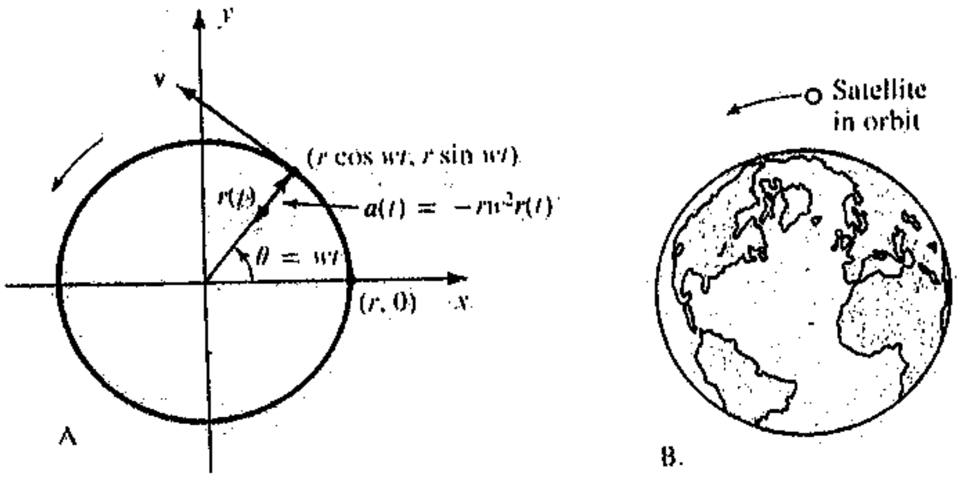


Figure 14-40. (A) Circular motion with constant speed; centripetal acceleration points toward the Centre. (B) Example 2, Satellite in orbit around the earth.

from the positive x – axis to the position vector r(t) is changing at a constant rate ω (the Greek letter *omega*); that is, $\theta = \omega t$. It follows that the position vector is

$$r(t) = r\cos \omega t\mathbf{i} + r\sin \omega t\mathbf{j}$$

Therefore,

$$v(t) = -r\omega \sin \omega t\mathbf{i} + r\omega \cos \omega t\mathbf{j}$$
$$a(t) = -r\omega^2 \cos \omega t\mathbf{i} + r\omega^2 \sin \omega t\mathbf{j}$$

Thus, $\mathbf{a}(t) = \omega^2\, \mathbf{r}(t)$, and this shows that **a** always points toward the Centre of the circle; its magnitude is $|a(t)| = r\omega^2$. Since the constant speed is $v = |v(t)| = r\omega$, we also have the important relationship $v^2 = r|a(t)|$ or

$$|a(t)| = \frac{v^2}{r}$$

This holds for all circular motion with constant speed.

Example 2

(*Satellite in orbit).* Suppose that a satellite is in circular orbit 200 miles above the earth. What is its speed and period? (Assume the radius of the earth is 4,000 mi and the acceleration due to gravity is 32 ft/sec^2.)

Solution

The radius of the circular path is r = 4,200 mi; the acceleration vector points toward the Centre of the earth and its magnitude is 32ft/sec^2. To find the speed, we use (14.43) making sure the units of measurement are compatible.

$v^2 = r|a(t)|$ = (4,200 mi)(32 ft/sec^2)(5,280 ft/mi)
$\approx$7.1 x 10^8 ft^2/sec^2

Taking square roots, we have

v $\approx$ 2.7 x 10^4 ft/sec (about 18,409 mph)

The period is the time it takes for one revolution.

period = $\dfrac{2\pi r}{speed} = \dfrac{2\pi(4{,}200mi)(5{,}280\,ft/mi)}{2.7x10^4\ ft/\sec}$
$\approx$5.161 seconds (about 86 minutes)

3.2 Force and Motion

Suppose a particle has constant mass *m*. Then Newton's second law of motion states that the product of *m* and the acceleration **a** of the particle equals the total external force acting on the particle.

14.44 **F** = *m***a** (force = mass x acceleration)

If the force is a given function of time, and the initial velocity and initial position are known, then it is possible to obtain the path of the particle by integration.

Example 3

The force acting on a particle at time *t* is $\mathbf{F}(t) = 6t\mathbf{i} + \mathbf{j}$. If the particle starts from the point (3,-1,2) with the velocity v(0) = 4**k**, find parametric equations of its path.

Solution

The path is obtained by finding the position vector function r(*t*). Since $\mathbf{F} = m\mathbf{a} = m\mathbf{r}''$, we start this problem with the equation

$$\mathbf{r}''(t) = \frac{1}{m}\mathbf{F} = \frac{1}{m}(6t\mathbf{i} + \mathbf{j})$$

Integration of both sides yields

(14.45) $(t) = r'(t) = \dfrac{1}{m}(3t^2\mathbf{i} + t\mathbf{j}) + C$

We are given that v(0) = 4k; thus, C = 4k and (14.45) becomes

$$r'(t) = \frac{1}{m}(3t^2\mathbf{i} + t\mathbf{j}) + 4\mathbf{k}$$

Integrate once again;

$$\mathbf{r}(t) = \frac{1}{m}\left(t^3\mathbf{i} + \frac{t^2}{2}\mathbf{j} + 4t\mathbf{k} + \mathbf{C}\right.$$

The starting point (3, -1, 2) yields r(0) = 3**i** – **j** + 2**k** = **C**. Therefore,

$$\mathbf{r}(t) = \left[\frac{1}{m}\left(t^3 i + \frac{t^2}{2} j\right) + 4tk\right] + [3\mathbf{i} - \mathbf{j} - 2\mathbf{k}]$$

$$= \left(\frac{t^3}{m} + 3\right)\mathbf{i} + \left(\frac{t^2}{2m} - 1\right)\mathbf{j} + (4t + 2)\mathbf{k}$$

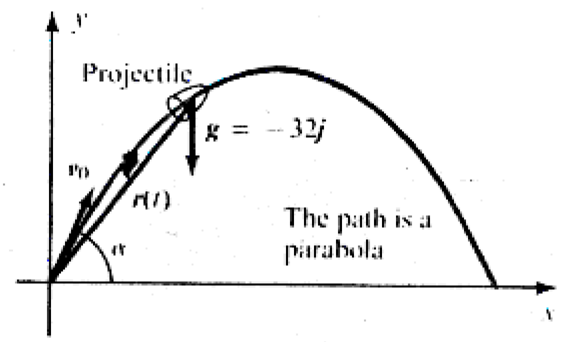
It follows that parametric equations of the path are

$$x = \frac{t^2}{m} + 3 \quad y = \frac{t^2}{2m} - 1 \quad z = 4t + 2 \quad t \geq 0$$

The method of Example 3 can be applied to objects in motion near the surface of the earth. If air resistance is neglected, such objects are subject only to the force of gravity **F** = *m*g, which is constant. In this case, the action takes place in the plane determined by **g** and the initial velocity vector $\mathbf{v}_0$. This is the situation for the motion of a projectile; that is, an object launched into the air and allowed to move freely. The plane of motion is taken to be the xy-plane and the acceleration due to gravity is

$$g = 32\mathbf{j}$$

which points straight down.



Example 4

(*Path of a projectile*). A projectile is launched from the origin with an initial speed of v_0 ft/sec at an angle α from the horizontal (Figure 14-41); that is, the initial velocity vector is $v_0 = v_0 \cos \alpha i + v_0 \sin \alpha j$. Show that the path is part of a parabola.

Solution

The acceleration g in this case is known; it points straight down and its magnitude is always 32 ft/sec^2. Therefore,

$$\mathbf{r}''(t) = g = -32\mathbf{j}$$

Integration of both sides yields

(14.46) $\quad v(t) = \mathbf{r}'(t) = -32t\mathbf{j} + \mathbf{C}$

Since $v(0) = v_0 = C$, we have

$$C = v_0 \cos \alpha\mathbf{i} + v_0 \sin \alpha\mathbf{j}$$

and (14.46) becomes

$$\mathbf{r}'(t) = v_0 \cos \alpha\mathbf{i} + (v_0 \sin \alpha - 32t)\mathbf{j}$$

We integrate again to obtain

$$\mathbf{r}(t) = (v_0 \cos \alpha)\, t\mathbf{i} + [(v_0 \sin \alpha)t - 16t^2]\mathbf{j} + \mathbf{C}$$

Since the projectile starts from the origin, we have $r(0) = \mathbf{0} = C$; therefore,

$$\mathbf{r}(t) = (v_0 \cos \alpha)\, t\mathbf{i} + [(v_0 \sin \alpha)t - 16t^2]\mathbf{j}$$

Parametric equations for the path are

$x = (v_0 \cos \alpha)t$ and $y = (v_0 \sin \alpha)t - 16t^2$

To show that the path is a parabola, we solve the first equation for t, eliminate the parameter, and obtain

$$y = (\tan \alpha)x - \frac{16}{(v_0 \cos \alpha)^2} x^2$$

which is an equation of a parabola.

4.0 CONCLUSION

As in the summary.

5.0 SUMMARY

In summary; Circular Motion with Constant Speed, Force and Motion of particles are discussed and related examples were given and solved for the purpose of more understanding of the unit objectives.

6.0 TUTOR MARK ASSISGNMENT (TMA)

Find the velocity, speed, and acceleration of the following position vectors.

i. $\mathbf{r}(t) = \mathbf{i} - 2t\mathbf{j} + (t + 1)\mathbf{k}$
ii. $\mathbf{r}(t) = -3t\mathbf{i} + t\mathbf{j} + \mathbf{k}$
iii. $\mathbf{r}(t) = \cos t\mathbf{i} + \sinh t\mathbf{j} + t\mathbf{k}$

The acceleration and initial position and velocity of a particle are given. Find the position functions.

iv. $\mathbf{a}(t) = t\mathbf{i} - 6t\mathbf{j} + \mathbf{k}$; $r(0) = \mathbf{0}$, $v(0) = \mathbf{i}$
v. $\mathbf{a}(t) = 2t\mathbf{i} + t\mathbf{j} - 3\mathbf{k}$; $r(0) = \mathbf{i} + \mathbf{j}$, $v(0) = \mathbf{0}$

7.0 REFERENCES / FURTHER READING

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