MODULE 1 REVIEW OF VECTOR THEORY

UNIT 1 VECTOR ALGEBRA

CONTENTS

1.0 INTRODUCTION

The notion of vector has proved to be of greatest value in physics and mathematics. It is one of the most important concepts that would be studied in this course. It will be found to recur in a great variety of applications. A full appreciation of the value of vectors can come only after considerable experience with them. Two aspect of their usefulness worth emphasising are the following:

(1) Vectors enable one to reason about problems in space without use of co-ordinates axes. This is particularly true because the fundamental laws of physics do not depend on the particular position of co-ordinates axes in space. For example, the Newton's second law, that has the form:

$$F = m\underline{a}$$

Where F is the force vector and $\underline{a}$ is the acceleration vector of a moving particle of mass m. This does not necessarily depend on co-ordinate axis.

(2) Vector provides an economical "Short hand" for complicated formulas. For example, the condition that points P_1, P_2, P_3, and, P_4 lie in a plane can be written in the concise form as:

$$\underline{a} . \underline{b} \times \underline{c} = 0$$

Where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors represented by the directed segment, $\overrightarrow{P_1P_2}$

$\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_4}$ respectively. The significant of the dot (.) and cross ($\times$) will be explained later in this unit. The conciseness of vector formulae makes vector useful both for manipulation and understanding.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- discuss vectors and give its examples
- define unit vectors, rectangular vectors, and resolve vectors into components
- perform algebraic functions on vectors
- solve related problems on vectors.

3.0 MAIN CONTENT

3.1 Vector Algebra

3.1.1 Definitions

Definition 1: A vector in space is a combination of a magnitude (positive real number) and a direction.

A vector can be represented by a directed line segment $\overrightarrow{PQ}$ in space. It is convenient to represent vectors by bold letters such as **a, b, c,**…

Definition 2: Two vectors are said to be equal if their magnitude and directions are the same.

Definition 3: A zero vector is a vector whose magnitude is zero. We can represent zero vectors by a degenerated line segment $\overrightarrow{PP}$

3.1.2 Addition and Subtraction of Vectors

Given two vectors: **a, b,** then we can obtain a third vector **c = a + b** and if
b = c – a
This defines the operation of subtraction.

Addition and subtraction of vectors obey the following laws:
(1) **a + b = b + a** [Commutative law of addition]
(2) **a + (b + c) = (a + b) + c** [Associative law]

(3) **a + b = c if b = c – a**

(4) **a + 0 = a**

(5) **a – a =0**

Definition 4: If h is a number and **a** is a vector then, the expression h **a** is defined as vector whose magnitude is $|h|$.

Thus: $|ha| = |h|.|a|$

Two vectors **a, b,** are said to be collinear (or linearly dependent) if there are scalars $h_1, h_2,$ not both zero, such that

$$h_1 \mathbf{a} + h_2 \mathbf{b} = \mathbf{0}$$

This is equivalent to asserting that **a** and **b** are represented by parallel line segments.

Definition 5: Three vectors **a, b, c** are said to be coplanar (or linearly dependent) if there are scalar k_1, k_2, k_3 not all 0 such that:

$$k_1 \mathbf{a} + k_2 \mathbf{b} + k_3 \mathbf{c} = \mathbf{0}$$

In this case a**, b, c** can be represented by segments in the same plane. Let a and b be no collinear. Then every vector **c** coplanar with **a and b** can be represented in the form

$$c = k_1 a + k_2 b$$

For one and only one, choice of k_1, k_2.

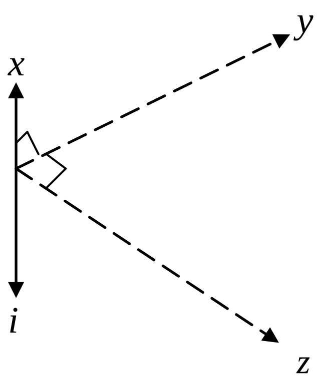
3.1.3 Unit Vector

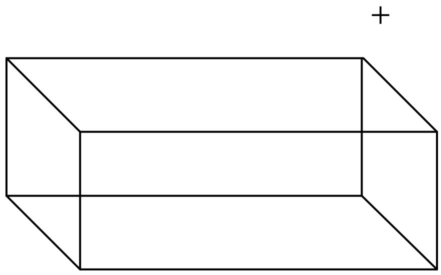
Unit vectors are vectors having unit length. Let a be any vector with length $|a| > 0$ then, $\frac{a}{|a|}$ is a unit vector denoted by $\hat{a}$ having the same direction as **a**

Then **a = a** $\hat{a}$

3.1.4 Rectangular Unit Vectors

The rectangular unit vectors $i.j, and, k$ are unit vectors having the direction of the positive x, y, and, z axes of a rectangular co-ordinates system. We use right-handed rectangular co-ordinate system unless otherwise specified.





3.1.5 Component of Vectors

Any vector in 3-dimensions can be represented with initial point at the origin 0 of rectangular co-ordinates systems.

Let (A_1, A_2, A_3) be the rectangular co-ordinates of the terminal point of A with initial point at 0. The vectors $A_1 i$, $A_2 j$ and $A_3 k$ are called the rectangular component vectors.

The sum of $A_1 i$, $A_2 j$ and $A_3 k$ i.e.

A= $A_1 i + A_2 j + A_3 k$ is a vector.

The magnitude of A is

$$|A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, if

$$r = xi + yj + zk$$

Then,

$$|r| = \sqrt{x^2 + y^2 + z^2}\ .$$

SELF-ASSESSMENT EXERCISE

i. Prove that for every four vectors x, y, z, and, w in space, scalars k_1, k_2, k_3, and, k_4
Not all 0, can be found such that
$k_1 x + k_2 y + k_3 z + k_4 w = 0$

ii. Let O, A, B be points of space. Show that the mid-point M of the segment $\vec{AB}$ is located by the vector $\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB})$

iii. Prove that the medians of a triangle intersect in a point which is a trisection point of each median.

4.0 CONCLUSION

In this unit, you have learnt about vectors, vector addition and subtraction. In addition, we also consider component of vectors and unit vectors as a special type of vectors. You are to read carefully and master every bit of the material in this unit for you to follow the material in the next unit.

5.0 SUMMARY

Recall that in this unit, we defined a vector as quantities having magnitude and directions. Two vectors are said to be equal if the directions and magnitudes are equal.

Also, we defined a unit vector as having magnitude equal to one. Finally, any vector in 3-dimension can be represented with initial point at the origin 0 of a rectangular co-ordinates systems. Thus, if (A_1, A_2, A_3) represent the rectangular co-ordinates of the terminal point of A then:

$A = A_1 i + A_2 j + A_3 k$ is a vector.

Magnitude of this vector A is define as

$|A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$ in particular if

$$r = xi + yj + zk$$

Then,

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that addition of vectors is commutative.

2. A car travels 3km due north, then 5 km northeast. Represent these displacements graphically and determine the resultant displacement by:

   Graphical method

   Analytical method.

If A, B, and, C are non-coplanar vectors and $x_1 A + y_1 B + z_1 C = x_2 A + y_2 B + z_2 C$, prove that it is necessary that $x_1 = x_2, y_1 = y_2, z_1 = z_2$.

3. Find the unit vector in the direction of the resultant of vectors $A = 2i - j + k$, $B = i + j + 2k$ $C = 3i - 2j + 4k$.

7.0 REFERENCES/FURTHER READING

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

UNIT 2 VECTOR ALGEBRA - PRODUCT OF VECTORS

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about product of vectors. We shall differentiate between scalar and vector products. These two concepts are very useful in vector analysis because many physical phenomena can be explained in terms of either scalar or vector products. For example, work done can be calculated as a scalar product of displacement and the applied force. This implies that if we let **F** represent force and **d** represent the displacement, then work done (**W)** can be defined as:

$$\mathbf{W} = \mathbf{F}.\mathbf{d}$$

Other physical interpretation of vector product will be discussed in this unit. You are advised to study this unit very carefully.

2.0 OBJECTIVES

At the end of this unit, you should be to:

- define scalar product of vectors and give examples
- define vector product and give examples
- solve accurately, all related exercises in this unit.

3.0 MAIN CONTENT

3.1 Types of Vector Products

Two types of vector products are recognised, namely:

1) Scalar Product

2) Vector Product

In what now follows, we shall define and explain scalar product of vectors.

3.2 Scalar Product

Let **a** and **b** be vectors then, the scalar product of **a** and **b** is defined as

$a.b = |a||b|\cos\theta$ ………………………………………………………… (1)

θ is the angle between them. The quantity $|b|\cos\theta$ which appears in (1) can be interpreted as the component of **b** in the direction of **a**. We can write it as

$comp_a^b = b\cos\theta$ ………………………………………………………… (2)

This component is a scalar which measures the length of the projection of **b** on a line parallel to **a.**

The notion of component is basic for application of vectors in mechanics. For example, the velocity vector or force vector can be described by giving its component in three mutually perpendicular directions. If a constant force F acts on an object moving from A to B along the segment $\vec{AB}$, the only component of F along AB does work. The work done is precisely the product of this component by the distance moved, thus:

Work = (force component in the direction of motion) . (distance)

Hence

Work = $F\cos\theta.|AB| = F.|AB|.$ ……………………………… (3)

Scalar product obeys the following laws:

(1) a.b = b.a (commutative)

(2) $a.(b+c) = a.b + a.c$ (Distributive law)

(3) $a.(kb) = (ka).b = k(a.b)$ Where k is a scalar.

We make the following inference from the scalar product of vectors.

(i) If $a.b = 0$ then a is perpendicular to b.

(ii) It is not permitted to cancel in an equation of the form

a.b = a.c and conclude that b = c.

For equation a.b = a.c it implies only that a.b = a.c = a.(b-c) = 0 that is a is perpendicular to b-c

We note that:

$i.i = 1, j.j = 1, and, k.k = 1$ and $i.j = 0, j.k = 0, and, k.i = 0$ .... (4)

Given that:

$a = a_1 i + a_2 j + a_3 k, and, b = b_1 i + b_2 j + b_3 k$ .... (5)

Then,

$a.b = (a_1 i + a_2 j + a_3 k).(b_1 i + b_2 j + b_3 k)$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$ .... (6)

SELF-ASSESSMENT EXERCISE

Show that:

$(A_1 i + A_2 j + A_3 k).(B_1 i + B_2 j + B_3 k) = A_1 B_1 + A_2 B_2 + A_3 B_3$

3.2.1 Direction Cosines

Recall from unit1 that if **a** is a vector of length 1 i.e. $|a| = 1$, then **a** will be termed a unit vector. In this case we denote **a** as:

$a = a_x i + a_y j + a_z k$

Then,
$a_x = a.1 = 1.1\cos\alpha = \cos\alpha.$

Where α is the angle between a, and, i. Where α is the angle between a and the positive x direction. In a similar manner

$a_y = \cos\beta, a_z = \cos\gamma$

Where β, and, γ are the angles between **a** and the $y, and, z,$ directions respectively.

From

a.b = $|a||b|\cos\theta$ then

$$\cos\theta = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

SELF-ASSESSMENT EXERCISE

Given that $u = i - j + k, v = i + j + 2k, w = 3i - k$

Evaluate: (a) $u + v + w, (b), 2u - v, (c), u.v$

3.3 The Vector Product

The vector product of a and b in that order is a vector $c = a \times b$ which is 0 if **a** and **b** are collinear and otherwise in such that:

$c = ab\sin\theta$, Where θ is the angle between a and b

The vector product satisfies the following laws:

(1) $a \times b = -(b \times a)$ (Anti- commutative law)

(2) $a \times (b + c) = a \times b + a \times c$ (Distributive law)

(3) $a \times (kb) = k(a \times b)$

(4) $a \times a = 0$

(5) $i \times j = k, j \times k = i, k \times i = j$

(6) $i \times i = 0, j \times j = 0, k \times k = 0$

(7) Let $\begin{array}{l} a = a_x i + a_y j + a_z k \\ b = b_x i + b_y j + b_z k \end{array}$

Then,

$$a \times b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$(a_y b_z - a_z b_y)i + (a_z b_x - a_x b_z)j + (a_x b_y - a_y b_x)k.$ ………………… (8)

Also you should note that:

$|a \times b| =$ area of parallelogram with sides **a** and **b**

SELF-ASSESSMENT EXERCISE

Given the vectors $a = 2i - j, \quad b = i + j + k \quad c = -2i + k$

Evaluate the following:

(i) $a \times b$ (ii) $c \times b$ (iii) $(a \times b) \times c$ (iv) $a.(a \times b)$ (v) $a \times (a \times b)$

3.4 Triple Product

In this section, we shall consider (1) The Scalar Triple Product (2) The Vector Triple Product.

3.4.1 The Scalar Triple Product

The scalar $a \times b.c$ is known as the scalar triple product a, b, c, in that order. We need to remark here that parentheses are not needed since $a \times (b.c)$ would have no meaning.

The scalar triple product satisfies the following laws:

(1) $a \times b.c = 0$ if and only if a,b,c, are coplanar

(2) $a \times b.c =$ volume of parallelepiped with edges a, b,and c

(3) $a \times b.c = a.b \times c.$

(4) $axb.c = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

(5) $a \times b.c = -b \times a.c = -b.a \times c$

SELF-ASSESSMENT EXERCISE

Evaluate the following:

(1) (i) $i.j \times k$ (ii) $(i + j).k + j$

(2) Given the vectors

$u = i - 2j + k$
$v = 3i + k$
$w = j - k$

Evaluate, (a) $u.v \times w$ (b) $w \times v.u$ (c) $(u+v).(v+w) \times w$

3.4.2 The Vector Triple Products

The expression $(a \times b) \times c$ and $a \times (b \times c)$ are known as vector triple products.

Note that the parentheses are necessary because for example;

$(i \times i) \times j = 0$ while $i \times (i \times j) = i \times k = -j$

The following identities are to be noted

(1) $a \times (b \times c) = (a.c)b - (a.b)c$

(2) $(a \times b) \times c = (c.a)b - (c.b)a$

We can prove the identity stated in (1) i.e.

$a \times (b \times c) = (a.c)b - (a.b)c$

Proof: Let
$$a = a_x i + a_y j + a_z k$$
$$b = b_x i + b_y j + b_z k$$
$$c = c_x i + c_y j + c_z k$$

Taking component i, then

$$i.a \times (b \times c) = \begin{vmatrix} 1 & 0 & 0 \\ a_x & a_y & a_z \\ \begin{vmatrix} b_y & b_z \\ c_c & c_z \end{vmatrix} & \begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} & \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \end{vmatrix}$$

$= a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z)$

$= b_x(a_x c_x + a_y c_y + a_x c_z) - c_x(a_x b_x + a_y b_y + a_z b_z)$

$= i.[(a.c)b - (a.b)c]$ … (9)

Prove the above for y and z components.

3.5 Axiomatic Approach to Vector Analysis

Recall from our previous section (unit 1 section 3.1.5) that we can represent a vector:

$r = xi + yj + zk$ is determined when its components (x, y, z) relative to some co-ordinate system are known. In adopting an axiomatic approach it is natural for us to make the following:

Definition: A 3-dimensional vector is an ordered triplet of real numbers (A_1, A_2, A_3).

With the above definition, we can define equality, vector addition and subtraction, etc.

Let $A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ then

1. A= B if and only if $A_1 = B_1, A_2 = B_2, A_3 = B_3$
2. A+ B = $(A_1 + B_1, A_2 + B_2, A_3 + B_3)$
3. A - B = $(A_1 - B_1, A_2 - B_2, A_3 - B_3)$
4. 0 = (0, 0, 0)
5. $mA = m(A_1, A_2, A_3) = (mA_1, mA_2, mA_3)$
6. A.B = $A_1.B_1 + A_2.B_2 + A_3.B_3$
7. Length or magnitude of A=$|A| = \sqrt{A.A} = \sqrt{A_1^2 + A_2^2 + A_3^3}$

From these, we obtain other properties of vectors, such as $A + B = B + A,$ $(A + B) + C = A + (B + C), A.(B + C) = A.B + A.C.$ By defining the unit vectors:

$i = (1,0,0) \quad j = (0,1,0) \quad k = (0,0,1)$

We can show that:

$A = A_1 i + A_2 j + A_3 k$

In like manner we can define $A \times B = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1)$

After this axiomatic approach has been developed, we can interpret the result geometrically or physically. For example we can show that A.B= $AB\cos\theta$

and $|A \times B| = AB\sin\theta$

4.0 CONCLUSION

In this unit, you have learnt about scalar multiplication and cross multiplication of vectors. We have also considered scalar and vector triple products. The application of these concepts will be apparent as we proceed further in this course.

5.0 SUMMARY

In summary, we recap the following about vector products, namely:

- Given that A and B are vectors, then the scalar product of A and B is defined as, A.B=$|A||B|\cos\theta$
- If A= $A_1 i + A_2 j + A_3 k$, and B= $B_1 i + B_2 j + B_3 k$ then A.B = $A_1 B_1 + A_2 B_2 + A_3 B_3$
- If A.B = 0 and A and B are not null vector, then A and B are perpendicular.
- Also $A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$
- $|A \times B| =$ the area of parallelogram with sides A and B.
- If $A \times B = 0$ and A and B are not null vectors, then A and B are parallel.
- $A \times B = -B \times A$

We also note the following about triple products of vectors. Dot and cross multiplication of three vectors A, B and C may produce meaningful products of the form (A.B) C, A.($B \times C$) and $A \times (B \times C)$.

The following laws are valid:

- (A.B)C$\neq$ A(B.C) in general
- $A \times (B \times C) \neq (A \times B) \times C$
- $A \times (B \times C) = (A.C)B - (A.B)C$
- $(A \times B) \times C = (A.C)B - (B.C)A$

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that $A.(B + C) = A.B + A.C$
2. Evaluate $|(A + B).(A - B)|$ if $A = 2i - 3j + 5k$ and $B = 3i + j - 2k$
3. Find the unit vector perpendicular to the plane of the vectors A=3i-2j+4k and B= i+j-2k
4. Given that A=2i+j-3k, B=i-2j+k, C=-i+j-4k, then find (a) A. (B$\times C$) (b) C. (A$\times B$)

7.0 REFERENCES/FURTHER READING

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

UNIT 3 VECTOR FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about vector functions. You will also learn about limit and continuity of vector functions. You will find derivatives of vectors and this will allow you to determine vector velocity. Finally, we shall give geometric interpretation to vector derivatives.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define limit and continuity of vector functions
- examine the derivatives of vector functions
- give a geometric interpretation to vector derivatives and be able to determine vector velocity
- solve all related problems on vector functions.

3.0 MAIN CONTENT

3.1 Vector Function of One Variable

Given an interval $t_1 \le t \le t_2$, suppose we assign a vector u in space, then u is said to be given a vector function of t over that interval.

For example,

$u = t^2 i + t^3 j + \sin tk$

Where i, j, k form a triple of mutually perpendicular unit vectors. If a co-ordinate system is chosen in space then the vector u can always be expressed in the form

$u = u_x i + u_y j + u_z k$

Where $u_x, u_y, and_, u_z$ are the corresponding components. These components themselves depend on *t*.

Suppose the axes are fixed and independent of ***t***, then we can write

$u_x = f(t)$, $u_y = g(t)$ and $u_z = h(t)$, $t_1 \le t \le t_2$

Thus, a vector functions of t, determines three scalar functions of *t*. Conversely, if $f(t), g(t)$ and $h(t)$ are three scalar functions of t defined on the interval, $t_1 \le t \le t_2$ then the vector *u* is given as

$u = f(t)i + g(t)j + h(t)k$ This is a vector function of *t*.

3.2 Limit and Continuity of Vector Function

The vector function $u = u(t)$ is said to have a limit v as t approaches t_o. This implies that $\lim_{t \to t_o} u(t) = v$, if $|u(t) - v| < \varepsilon$ whenever $|t - t_o| < \delta$.

The implication of this is that the difference between $u(t)$ and v can be made arbitrarily small for *t* sufficiently close to t_0

Continuity: The function $u = u(t)$ is said to be continuous at the value t_0 if one has

$$\lim_{t \to t_0} u(t) = u(t_0)$$

We can establish by prove that $u(t)$ is continuous at a value t_o, if and only if its component u_x, u_y, and, u_z are all continuous.

Also given two vectors $u_1(t), and, u_2(t)$ such that they are both continuous functions for $t_1 \le t \le t_2$ then the functions:

$u_1(t) + u_2(t)$, $u_1(t).u_2(t)$ and $u_1(t) \times u_2(t)$ are continuous functions of t over the defined interval.

3.3 Derivative of a Vector Function

Velocity Vector: The derivative of the vector function $u = u(t)$ is defined as a limit.

$$\frac{du}{dt} = \lim_{\Delta t \to 0} \frac{u(t + \Delta t) - u(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t}$$

We can define the above in terms of component as follows:

$$u(t+\Delta t)-u(t)=[f(t+\Delta t)-f(t)]i+[g(t+\Delta t)-g(t)]j+[h(t+\Delta t)-h(t)]k$$

Hence, on dividing by Δt and letting $\Delta t \to 0$ one finds

$$\frac{du}{dt}=f'(t)i+g'(t)j+h'(t)k$$

$$=\frac{du_x}{dt}i+\frac{du_y}{dt}j+\frac{du_z}{dt}k$$

Therefore, to differentiate a vector function, each component must be differentiated separately.

3.4 Geometric Interpretation

Let S be the distance traversed by P from $t=t_1$ up to time t, then

$$\frac{ds}{dt}=\sqrt{f'(t)^2+g'(t)^2+h'(t)^2}$$

$$=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}$$

Let $u=\overrightarrow{OP}$ the position vector of the moving point P, then the vector

$v=(d/dt)\overrightarrow{OP}$ is the tangent to the curve traced by P and has at each point a magnitude

$$|v|=\left|\frac{du}{dt}\right|=\sqrt{f'(t)^2+g'(t)^2+h'(+)^2}$$

The conclusion drawn from above is that v is precisely the velocity vector of the moving point P for v is the tangent to the path and has magnitude $v=ds/dt$ (speed) and clearly points in the direction of motion.

We then have the following rule:

$\frac{d}{dt}\overrightarrow{OP}=$ velocity of P, where 0 is a fixed reference point.

Finally we consider the following differentials: Given that

$A(x,y,z) = A_1(x,y,z)i + A_2(x,y,z)j + A_3(x,y,z)k$

Then,

$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz$, is the differential of A.

Remarks: Derivatives of products obey rules similar to those for scalar functions. However when cross product are involved, the order may be important. Some examples are:

(a) $\frac{d}{dx}(\phi A) = \phi\frac{dA}{dx} + \frac{d\phi}{dx}A$

(b) $\frac{\partial}{\partial x}(A.B) = A.\frac{\partial B}{\partial x} + \frac{\partial A}{\partial x}.B$

(c) $\frac{\partial}{\partial z}(A \times B) = A \times \frac{\partial B}{\partial z} + \frac{\partial A}{\partial z} \times B$

Solved Problems

1. Suppose $u = r\cos(wt)i + r\sin(wt)j$ where r and w are constants. Let the point P moves according to the equations $x = r\cos(wt),\ y = r\sin(wt)$ which represent the circle $x^2 + y^2 = r^2$ in the xy-plane. The polar angle θ of P at time t is $\theta = wt$. Find the angular velocity, the vector velocity and the speed of the movement.

Solution:

The angular velocity of P

$= \frac{d\theta}{dt} = w$

2. Vector velocity

$v = \frac{dv}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j = -rw\sin(wt)i + rw\cos(wt)j$

3. Speed is

$\frac{ds}{dt} = \sqrt{r^2w^2\sin^2(wt)i + r^2w^2\cos^2(wt)} \quad = \quad rw, w \geq 0$

Problem 2: If $r = (t^3 + 2t)i - 3e^{-2t}j + 2\sin 5tk$, Find (a) $\frac{dr}{dt}$ (b) $\left|\frac{dr}{dt}\right|$, (c) $\frac{d^2r}{dt^2}$

(d) $\left|\dfrac{d^2r}{dt^2}\right|$, at t=0 and give a possible physical significance.

Solution:

(a) $\dfrac{d}{dt}(t^3+2t)i + \dfrac{d}{dt}(-3e^{-2t})j + \dfrac{d}{dt}(2\sin 5t)k$

= $3t^2+2)i + 6e^{-2t}j + 10\cos 5tk$

At t=0 $dr/dt = 2i + 6j + 10k$

(b) From (a) $|dr/dt| = \sqrt{(2)^2+(6)^2+(10)^2} = \sqrt{140} = 2\sqrt{35}$ at t=0.

(c) $\dfrac{d^2r}{dt^2} = \dfrac{d}{dt}(\dfrac{dr}{dt}) = \dfrac{d}{dt}\{(3t^2+2)i + 6e^{-2t}j + 10\cos 5tk) = 6ti - 12e^{-et}j - 50\sin 5tk$

At t=0, $d^2r/dt^2 = -12j$

(d) From (c) $|d^2r/dt^2| = 12$ at t=0.

If t represents time, these represent respectively the velocity, magnitude of the velocity, acceleration and magnitude of the acceleration at t=0 of a particle moving along the space curve $x = t^3 + 2t$ $y = -3e^{-2t}$, $z = 2\sin 5t$

4.0 CONCLUSION

In this unit, you have learnt about vector function, limit and continuity of vector functions, derivatives of vectors and geometrical interpretations of vector derivatives. In the next unit, we shall extend these derivatives into partial derivatives and apply the results in the orthogonal curvilinear co-ordinates.

5.0 SUMMARY

We now recap what you have learnt in this unit as follows:

- Given an interval $t_1 \le t \le t_2$, a vector function u can be assigned such that $u = u(t)$. For example, $u(t) = t^2 i + \sin tj + \cos^2 tk$ is a vector function of *t*.
- We can define a limit of the vector function as:
  $\lim u(t) = v$ if $|u(t) - v| < \varepsilon$ whenever $|t - t_0| < \delta$. This implies that the difference between $u(t)$ and v can be made arbitrarily small for *t* sufficiently close to t_0
- We define continuity of $u(t)$ as:
  $\lim_{t \to t_0} u(t) = u(t_0)$

If $u(t) = u_x(t)i + u_y(t)j + u_z(t)k$ then, we can prove that $u(t)$ is continuous if and only if all the components of $u(t)$ are continuous.

- We define derivatives of vectors as follows:

If $u(t) = u_x(t)i + u_y(t)j + u_z(t)k$ then

$$\frac{du(t)}{dt} = \frac{du_x(t)}{dt} i + \frac{du_y}{dt} j + \frac{du_z}{dt} k$$

We also give a geometric interpretation of the derivatives of vectors.

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that $\frac{d}{du}(A.B) = A.\frac{dB}{du} + \frac{dA}{du}..B$ where A and B are differentiable functions of u.
2. If $A = x^2 \sin yi + z^2 \cos yj - xy^2 k,$ find dA
3. A particle moves along a space curve, $r = r(t),$ where $t,$ is the time measured from some initial time. If $v = |dr/dt| = ds/dt$ is the magnitude of the velocity of the particle (s is the arc length along the space curve measured from the initial position), prove that the acceleration ***a*** of the particle is given by:

$$a = \frac{dv}{dt}T + \frac{v^2}{\rho}N$$

Where T and N are unit tangent and normal vectors to the space curve and

$$\rho = \left|\frac{d^2 r}{ds^2}\right|^{-1} = \left\{\left(\frac{d^2 x}{ds^2}\right)^2 + \left(\frac{d^2 y}{ds^2}\right)^2 + \left(\frac{d^2 z}{ds^2}\right)^2\right\}^{-1/2}$$

4. Prove that grad $f(r) = \frac{f'(r)}{r} r,$ where $r = \sqrt{x^2 + y^2 + z^2}$ and $f'(r) = df/dr$ is assumed.

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

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