MODULE 2 DIFFERENTIAL OPERATORS

UNIT 1 THE OPERATOR DEL (∇)

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about certain differential operations which can be performed on scalar and vector fields. These operations have wide-ranging applications in the physical sciences. The most important operations are those of finding the gradient of a scalar field and the divergence and curl of a vector field. Central to all these differential operations is the vector operator ∇ which is called Del (or sometimes, nabla).

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the operator Del (∇)
- apply the operator in finding gradient of function $\phi(x, y, z)$
- give physical interpretation to gradient of $\phi(x, y, z)$
- solve correctly, exercises involving the use of gradient.

3.0 MAIN CONTENT

3.1 Operator Del (∇)

Consider the operator $\nabla(del)$ defined by:

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \qquad \text{……………… ……….. (1)}$$

Equation (1) is called operator Del. It has a lot of physical application in vector analysis as we shall see shortly.

If $\phi(x, y, z)$ and $A(x, y, z)$ have continuous first partial derivatives in a region, we can define the gradient of $\phi(x, y, z)$ as:

(1) Gradient: The gradient of $\phi(x, y, z)$ is defined by:

$$grad\phi = \nabla\phi = \frac{\partial\phi(x, y, z)}{\partial x}i + \frac{\partial\phi(x, y, z)}{\partial y}j + \frac{\partial\phi(x, y, z)}{\partial z}k$$

3.2 **Interpretation** of Gradient of $\phi(x, y, z)$

One interesting application of $grad\phi$ can be view as follows:

$\phi(x, y, z) = c$ ………………………………… (2)

Let equation (2) be equation of a surface then, $\nabla\phi$ is normal to this surface. To see this, let $\phi(x, y, z)$ be a scalar field.

Consider the differential defined by:

$dr = dxi + dyj + dzk$ ………………………… (3)

The corresponding differential in $\phi(x, y, z)$ is

$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$ ………………………….. (4)

$= \nabla\phi.dr$ ………………………….. (5)

Now if $\phi = c$ then $d\phi = 0$ therefore,

$\nabla\phi.dr = 0$ ………………………… (6)

Hence $\nabla\phi$ is normal to the surface given by the equation $\phi(x, y, z) = c$

Examples:

(1) Find the gradient of the scalar field $\phi = xy^2z^3$

Solution: $\nabla\phi = y^2z^3i + 2xyz^3j + 3xy^2z^3k$

(2) Given the function $\phi(x, y, z) = x^2y + yz$ at the point (1, 2,-1) find its rate of change with distance in the direction $a = i + 2j + 3k$. At this same point, what is the greatest possible rate of change with distance and in which direction does it occur?

Solution:

Gradient of ϕ is given by

$\nabla\phi = \nabla(x^2y + yz) = 2xyi + (x^2 + z)j + yk$

Now at the point (1, 2,-1), $\nabla\varphi = 4i + 2k$

The unit vector in the direction of a is $\hat{a} = \frac{1}{\sqrt{14}}(i + 2j + 3k)$, so the rate of change of ϕ with distance s in this direction is

$$\frac{d\phi}{ds} = \nabla\phi.\hat{a} = \frac{1}{\sqrt{14}}(4+6) = \frac{10}{\sqrt{14}}$$

From the above discussion, at the point (1, 2,-1), $d\phi/ds$ will be greatest in the direction of $\nabla\phi = 4i + 2k$ and has the value $|\nabla\phi| = \sqrt{20}$ in this direction.

The gradient obeys the following laws:

$$grad(f + g) = gradf + gradg$$
$$grad(fg) = fgradg + ggradf$$

In addition to these, we note that the gradient operation also obey the chain rule as in ordinary differential calculus, i.e. if ϕ and φ are scalar fields in region R, then

$$\nabla[\phi(\varphi)] = \frac{\partial\phi}{\partial\varphi}\nabla\varphi \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

4.0 CONCLUSION

In this unit, you have learnt about gradient of vector and scalar fields. In the next unit, we shall examine divergence of a vector field and how it relies on the operator Del. It is very important for you to learn this operator very well before you make any meaningful progress beyond this point.

5.0 SUMMARY

You have learnt the following in this unit:

- The operation Del (∇) is defined as
  $$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$
- If $\phi(x, y, z)$ is a scalar field then the gradient of $\phi(x, y, z)$ is defined as
  $$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\,.$$
- The corresponding differential of $\phi(x, y, z)$ is given as
  $$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \;=\; \nabla\phi.dr$$

- Where,
  $dr = dxi + dyj + dzk$
- If $\phi(x, y, z) = c$ then, $d\phi = 0$ this implies that
  $\nabla\phi.dr = 0$, hence $\nabla\phi$ is normal to the surface given by $\phi(x, y, z) = c$

6.0 TUTOR-MARKED ASSIGNMENT

1. If $\phi = x^2 yz^3$ and A= $xzi - y^2 j + 2x^2 yk$ find (i) $\nabla\phi$ (ii) $\nabla.A$
2. Prove that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$ where c is a constant.
3. If $\phi = 2x^2 y - xz^3$ find $\nabla\phi$ and $\nabla^2\phi$

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H. J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

UNIT 2 DIVERGENCE OF A VECTOR FIELD

CONTENTS

1.0 INTRODUCTION

Divergence can be considered as a quantitative measure of how much a vector field diverges (spread out) or converges at any given point. For example, if we consider the vector field $v(x, y, z)$ describing the local velocity at any point in a fluid then the divergence is equal to the net rate of outflow of fluid per unit volume, evaluated at a point. We will be exposed to mathematical exposition of this very important concept in this unit. The prerequisite to our learning this unit is the thorough understanding of the unit 1 of this module.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain the divergence of a vector field
- explain the Laplacian
- solve the exercises at the end of this unit.

3.0 MAIN CONTENT

3.1 The Divergence of a Vector Field

Suppose we are given a vector field $v(x, y, z)$ in the domain D of space, given three scalar functions v_x, v_y, v_z. suppose these functions possess partial derivatives in D then the divergence is defined as:

$$div v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Formula (1) can be written in the symbolic form:

$div v = \nabla . v$ which implies:

$$\nabla.v = (\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k).(v_x i + v_y j + v_z k) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \text{...............} (2)$$

The divergence defined above has a physical significance. In fluid dynamics, it appears as a measure of the rate of decrease of density at a point. More precisely,

Let $u = u(x, y, z, t)$ denote the velocity vector of a fluid motion and let $\rho = \rho(x, y, z, t)$ denote the density.

Then $v = \rho u$ is a vector whose divergence satisfies the equation.

Then, $$divv = -\frac{\partial \rho}{\partial t} \quad \text{…………………} (3)$$

Equation (3) is called continuity equation of fluid mechanics. If fluid is incompressible, this reduce to the simpler equation

$$divu = 0 \quad \text{…………………} (4)$$

The divergence also plays an important role in the theory of electromagnetic fields. To see this, we note that the divergence of the electric force vector E satisfies the equation defined by:

$$divE = 4\pi\rho \quad \text{…………………} (5)$$

Where ρ is the charge density. Thus where there is no charge, equation (5) reduces to

$$divE = 0 \quad \text{………………….} (6)$$

The divergence has the following basic properties:

(1) *div* (*u*+*v*) =*div u*+*div v*

(2) $div(fu) = fdivu + gradf.u$ ……………......…. (7)

3.1.1 The Laplacian

Let $w = f(x, y, z)$ then the Laplacian of w is defined as

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \quad \text{……………………...…} (8)$$

The origin of the ∇^2 lies in the interpretation of ∇ as a vector differential operator defined before as:

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Symbolically,

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

If $z = f(x, y)$ and has second derivatives in the domain D and

$$\nabla^2 z = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

In the domain D, the z is said to be harmonic in D. We also used the same term for a function of three variables which has continuous second derivatives in a domain D in space and whose Laplacian is 0 in D. The two equations for harmonic functions:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

are known as the Laplacian equations in two and three dimensions respectively.

Remark: In the theory of elasticity, we have the following equation:

$$\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

The combination which appears above can be expressed in terms of the Laplacian as follows:

$$\nabla^2(\nabla^2 z) = \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

The expression in (14) is called biharmonic expression whose solutions are termed biharmonic functions. Harmonic functions arise in the theory of electromagnetic fields, in fluid dynamics, in the theory of heat conduction, and many other parts of physics.

3.2 Illustrative Examples

1) Given that A= $xzi - y^2 j + 2x^2 yk$, find the divergence of A.

Solution: The divergence of A is defined as

$$\nabla.A = \begin{aligned} &(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k).(xzi - y^2 j + 2x^2 yk) \\ &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2 y) = z - 2y \end{aligned}$$

2) Prove that $\nabla.(\phi A) = (\nabla\phi).A + \phi(\nabla.A)$

Solution: $\nabla.(\phi A) = \nabla.(\phi A_1 i + \phi A_2 j + \phi A_3 k)$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial\phi}{\partial x}A_1 + \frac{\partial\phi}{\partial y}A_2 + \frac{\partial\phi}{\partial z}A_3 + \phi(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z})$$

$$= (\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k).(A_1 i + A_2 j + A_3 k) + \phi(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k).(A_1 i + A_2 j + A_3 k)$$

$= (\nabla\phi).A + \phi(\nabla.A)$

3). Given that $\phi = 2x^2y - xz^3$ find $\nabla^2\phi$

Solution: $\nabla^2\phi =$ Laplacian of $\phi = \nabla.\nabla\phi = \frac{\partial}{\partial x}(4xy - x^2) + \frac{\partial}{\partial y}(2x^2) + \frac{\partial}{\partial z}(-3xz^2)$

$$= 4y - 6xz$$

4.0 CONCLUSION

In this unit, you have learnt about divergence of vector field, you have also learnt about Laplacian and discussed various applications of these concepts to physical phenomenal. You are advised to read this unit properly and carefully, before moving to other unit.

5.0 SUMMARY

It should be noted that divergence is a measure of how much a vector field spread out or converges.

If $v(x, y, z)$ is a vector field, then its divergence is defined as

$$divv = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

We may derive from the definition of divergence and also define Laplacian as follows:

$$\nabla.(gradf) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We also considered other physical application such as application of biharmonic functions of the form

$\nabla^2(\nabla^2 z) = \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial^2 x\partial^2 y} + \frac{\partial^4 z}{\partial y^4}$ in the theory of elasticity.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given that the vector field $v = 2xi + yj - 3zk,$ verify that the divergence of v (*div v*) is zero.
2. Evaluate $[(xi - yj).\nabla](x^2 i - y^2 j + z^2 k)$
3. Given that $\phi = x^2 yz^3$ and A= $xzi - y^2 j + 2x^2 yk$. Evaluate $div(\phi A)$

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H. J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

UNIT 3 THE CURL OF A VECTOR FIELD

CONTENTS

1.0 INTRODUCTION

In this unit, we will learn about curl of a vector field. This concept has a wide range of application in physical phenomenal such as electromagnetic theory. Those concepts we learnt earlier such as gradient of vector field and divergence theory will be applied later in the theory of orthogonal curvilinear co-ordinates systems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define curl of vector field correctly
- interpret the physical implication of curl of vector field
- solve all the associated mathematical problems involving the curl of vector fields.

3.0 MAIN CONTENT

3.1 The Curl of a Vector Field

We can define the curl of a vector field as follows:

Let $v(x, y, z)$ be a vector field then, the curl of vector $v(x, y, z)$ is

Curl v= $\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$ .... (1)

Equation (1) can be expressed as:

Curl v = $(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})i + (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}) j + (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})k$ .... (2)

This vector field has a meaning independent of the choice of axes. We shall see this in the treatment of orthogonal curvilinear co-ordinates to be considered in the next module.

The curl of vector field is important in the analysis of the velocity field of fluid dynamics and in the analysis of electromagnetic force fields. For example, curl can be interpreted as measuring angular motion of a fluid and the condition is:

$$\text{Curl v} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

For a velocity field v characterises what are termed irrotational flows. The analogous equation is given as:

$$\text{Curl E} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

For the electric force vector E, it holds when only electrostatic forces are present.

Recall that: if $\nabla \times V = 0$ in a region, we say that the flow is irrotational in that region. The implication of this is that the circulation around a closed curve in a simple region where the flow is irrotational is zero. If the fluid is incompressible and there is no distribution of sources or sink in the region, we have also $\nabla . V = 0.$ since the condition $\nabla \times V = 0$ implies the existence of a potential φ such that

$$V = \nabla \varphi \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

We see that if also $\nabla . V = 0$ then it follows that $\nabla . \nabla \varphi = \nabla^2 \varphi = 0.$ That is, in the flow of an incompressible irrotational fluid without distributed sources or sinks the velocity vector is the gradient of a potential φ which satisfies the equation

$$\nabla^2 \varphi = 0 \ \text{ or } \ \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x^2} = 0 \ \ldots\ldots (6)$$

Equation (6) is known as Laplace's equation already discussed in (Unit 2, Module 2)

Generally, in any continuously differentiable vector field F with zero divergence and curl in a simple region, the vector F is the gradient of a solution of Laplace's equation.

Solutions of this equation are called harmonic functions.

3.2 Illustrative Examples

1) If A = $xz^3 i - 2x^2 yzj + 2yz^4 k$. Find $\nabla \times A$ (or curl A) at the point (1,-1, 1)

Solution:

$$\nabla \times A = (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k) \times (xz^3 i - 2x^2 yzj + 2yz^4 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2 yz & 2yz^4 \end{vmatrix}$$

$$= [\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2 yz)]i + [\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)]j + [\frac{\partial}{\partial x}(-2x^2 yz - \frac{\partial}{\partial y}(xz^3)]k$$

$= (2z^4 + 2x^2 y)i + 3xz^2 j - 4xyzk = 3j + 4k$ at point (1,-1, 1)

2) If $A = x^2 yi - 2xzj + 2yzk$ find $CurlCurlA$

Solution:

$$curlcurlA = \nabla \times (\nabla \times A)$$

$$= \nabla \times \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix} = \nabla \times [(2x + 2z)i - (x^2 + 2z)k]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix} = (2x + 2)j$$

3) Prove that $\nabla \times (\nabla \phi) = 0$

Solution:

$$\nabla \times (\nabla \phi) = \nabla \times (\frac{\partial \phi}{\partial x} i - + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= [(\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial z}) - \frac{\partial}{\partial z}(\frac{\partial\phi}{\partial y})]i + [\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial x}) - \frac{\partial}{\partial x}(\frac{\partial\phi}{\partial z})]j + [\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial y}) - \frac{\partial}{\partial y}(\frac{\partial\phi}{\partial x})]k$$

$$= (\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y})i + (\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z})j + (\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x})k = 0$$

This is only true when ϕ is continuously differentiable, hence the order of the differentiation is immaterial.

4.0 CONCLUSION

In this unit you have learnt about curl and various applications of curl to physical situations. Study this unit carefully before moving to the next unit of this course.

5.0 SUMMARY

We recall that in this unit we defined a curl of a vector field, as

$$\text{Curl} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

You are required to master this formula properly because of its physical application as we proceed in studying this course.

6.0 TUTOR-MARKED ASSIGNMENT

Obtain the curls of the following vectors:

i. xi, *ii.* r, *iii.* $(xi - yj)/(x + y)$ *iv.* $i\sin y + jx(1 + \cos y)$

If curlA = 0 where A= $(xyz)^m(x^n i + y^n j + z^n k)$ show that either m = 0 or n = -1

If v = r (a.r) where a is a constant vector show that

Curlv = $a \wedge r$ (ii) curl $(a \wedge r) = 2a$

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H. J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.