MODULE 3 ORTHOGONAL CURVILINEAR CO-ORDINATES

UNIT 1 JACOBIANS

CONTENTS

1.0 INTRODUCTION

A useful tool for the operation on orthogonal curvilinear co-ordinate systems is the Jacobian. Since most of the co-ordinate systems are different from the Cartesian co-ordinate some transformations are usually required which will necessitate the need to find the scale factors of these transformation, in doing this we may need to find the Jacobian of the transformation before we can be able to find the required scale factors.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define Jacobian
- solve exercises involving the use of Jacobian.

3.0 MAIN CONTENT

3.1 Definition of Jacobian

The Jacobian of x and y for two independent variables m and n is the determinant

$$\begin{vmatrix} (\frac{\partial x}{\partial m})_n & (\frac{\partial x}{\partial n})_m \\ (\frac{\partial y}{\partial m})_n & (\frac{\partial y}{\partial n})_m \end{vmatrix}$$

Where $x = f_1(m,n)$ and $y = f_2(m,n)$.

The customary notation is

$$\frac{\partial(x,y)}{\partial(m,n)} = \begin{vmatrix} (\frac{\partial x}{\partial m})_n & (\frac{\partial x}{\partial n})_m \\ (\frac{\partial y}{\partial m})_n & (\frac{\partial y}{\partial n})_m \end{vmatrix} \qquad \text{…………..} \quad (1)$$

It is obvious from (1) that

$$\frac{\partial(x,y)}{\partial(m,n)} = (\frac{\partial x}{\partial m})_n(\frac{\partial y}{\partial n})_m - (\frac{\partial x}{\partial n})_m(\frac{\partial y}{\partial m})_n \qquad \text{…………...} \quad (2)$$

3.1.1 Properties of Jacobians

Jacobians have the following basic properties:

(a) We note from (1) that

$$\frac{\partial(y,x)}{\partial(m,n)} = \begin{vmatrix} (\frac{\partial y}{\partial m})_n & (\frac{\partial y}{\partial n})_m \\ (\frac{\partial x}{\partial m})_n & (\frac{\partial x}{\partial n})_m \end{vmatrix} \qquad \text{…..............} \quad (3)$$

Therefore,

$$\frac{\partial(y,x)}{\partial(m,n)} = \left(\frac{\partial y}{\partial m}\right)_n\left(\frac{\partial x}{\partial n}\right)_m - \left(\frac{\partial y}{\partial n}\right)_m\left(\frac{\partial x}{\partial m}\right)_n \qquad \text{……………} \quad (4)$$

By comparing (2) and (3), we could infer that

$$\frac{\partial(y,x)}{\partial(m,n)} = -\frac{\partial(x,y)}{\partial(m,n)} \qquad \text{…………..} \quad (5)$$

(b) Similarly, according to (1)

$$\frac{\partial(y,z)}{\partial(x,z)} = \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_z & \left(\frac{\partial y}{\partial z}\right)_x \\ \left(\frac{\partial z}{\partial x}\right)_z & \left(\frac{\partial z}{\partial z}\right)_x \end{vmatrix} \qquad \text{……………....…} \quad (6)$$

$$\frac{\partial(y,z)}{\partial(x,z)} = \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_z & \left(\frac{\partial y}{\partial z}\right)_x \\ 0 & 1 \end{vmatrix} \qquad \text{……………..…… (7)}$$

We see that:

$$\frac{\partial(y,z)}{\partial(x,z)} = \left(\frac{\partial y}{\partial x}\right)_z \qquad \text{……………..…… (8)}$$

From (8) it is obvious that all partial derivatives can be represented by Jacobians.

(c) It is easy to note that

$$\frac{\partial(y,x)}{\partial(a,b)}\frac{\partial(a,b)}{\partial(m,n)} = \frac{\partial(y,x)}{\partial(m,n)} \qquad \text{……………..…… (9)}$$

(d) From equation (1) it follows that

$\frac{\partial(m,n)}{\partial(m,n)} = 1, \quad \frac{\partial(x,x)}{\partial(m,n)} = 0$ and if k is constant, then

$$\frac{\partial(k,x)}{\partial(m,n)} = 0 \qquad \text{……………..…… (10)}$$

It is possible using equations (8), (9) and (10) to transform partial derivatives.

To see this, consider the quantity defined by:

$\left(\frac{\partial T}{\partial p}\right)$, which we can express as

$$\left(\frac{\partial T}{\partial p}\right) = \frac{\partial(T,s)}{\partial(p,s)} \qquad \text{……………..…… (11)}$$

While

$$\frac{\partial(s,T)}{\partial(p,s)} = \frac{\partial(s,T)}{\partial(p,T)} / \frac{\partial(p,s)}{\partial(p,T)} \qquad \text{……………..…… (12)}$$

This is in conformity with equation (11) we note that

$$\frac{\partial(s,T)}{\partial(p,T)} = \left(\frac{\partial s}{\partial p}\right)_T \qquad \text{……………..…… (13)}$$

We may write Jacobian in a notational form as follows:

To find the Jacobian of the function $u(x, y.z), v(x, y, z), w(x, y, z)$, we express it as:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J(\frac{u,v,w}{x,y,z}) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

We should also note that in general:

$$\frac{\partial(x_1, x_2, \ldots\ldots x_n)}{\partial(u_{1,}\, u_2, \ldots\ldots. u_n).} \cdot \frac{\partial(u_1, u_2, \ldots\ldots. u_n)}{\partial(x_1, x_2, \ldots\ldots. x_n)} = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

SELF-ASSESSMENT EXERCISE

1. Consider the two functions defined as:
   $u_1 = ax + by + c$
   $u_2 = dx + ey + f$

   Investigate weather they are functionally dependent.
2. If u and v are functions of r and s, and r and s are also functions of x and y, prove that :

   $$\frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

3.2 Jacobian and Curvilinear Co-ordinates: Change of Variables in Integrals

Given the equations:

$$x = x(u_1, u_2, u_3), y = y(u_1, u_2, u_3), z = z(u_1, u_2, u_3) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

which defines curvilinear co-ordinates, u_1, u_2, and u_3 in space.

Suppose we write:

$$U_k = i\frac{\partial x}{\partial u_k} + j\frac{\partial y}{\partial u_k} + k\frac{\partial z}{\partial u_k} \qquad \text{(k=1, 2, 3)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Then for u_1, u_2, u_3 the volume element in the new co-ordinate is given as

$$d\tau = (U_1, U_2, U_3)du_1du_2du_3 \qquad \text{……………….......… (18)}$$

If the co-ordinates are so ordered that the right –hand member is positive. Now we define

$$U_1.U_2 \times U_3 = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_3} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u_1, u_2 u_3)} \qquad \text{……….......…… (19)}$$

Now since the determinant is unchanged if the row and column are interchanged then we may write

$$d\tau = \left| \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \right| du_1du_2du_3 \qquad \text{…………………… (20)}$$

We now present the change of variable formula as

$$\iiint_R w(x, y, z)dxdydz = \iiint_{R^*} W(u_1, u_2, u_3) \left| \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \right| du_1du_2du_3 \qquad \text{…...……(21)}$$

Where

$W(u_1, u_2, u_3) = w[x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3)]$, and R^* is the u_1, u_2, u_3 region into which we transform the $x, y, z,$ region R.

The Jacobian $\partial(x, y, z)/\partial(u_1, u_2, u_3)$ is continuous and nonzero in R^*

If we are given equations in two dimensions such as

$$x = x(u_1, u_2), \quad y = y(u_1, u_2) \qquad \text{………………………… (22)}$$

Note that (22) can be interpreted as defining curvilinear co-ordinates in the $xy - plane$.

The vectors:

$$U_1 = i\frac{\partial x}{\partial u_1} + j\frac{\partial y}{\partial u_1}, \quad U_2 = i\frac{\partial x}{\partial u_2} + j\frac{\partial y}{\partial u_2} \qquad \text{……………….......… (23)}$$

are the tangent to the co-ordinate curves, with the lengths ds_1/du_1 and ds_2/du_2

The vector element of plane is then given by

$$dA = (U_1 \times U_2)du_1 du_2 = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & 0 \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & 0 \end{vmatrix} du_1 du_2$$

This relation gives the result

$$dA = |dA| = \left|\frac{\partial(x.y)}{\partial(u_1, u_2)}\right| du_1 du_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(24)}$$

Hence,

$$\iint_D w(x,y)dxdy = \iint_D W(u_1,u_2)\left|\frac{\partial(x,y)}{\partial(u_1,u_2)}\right| du_1 du_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(25)}$$

4.0 CONCLUSION

In this unit, we have defined Jacobians as a preparatory for us to study curvilinear co-ordinate systems.

5.0 SUMMARY

Recall that, we studied Jacobian as a useful tool for determining transformation from one space to another; you are to read and understand this unit carefully so that you will be able to understand the content of the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. The transformation from rectangular to cylindrical co-ordinates is defined by the transformation:

   $x = \rho\cos\phi, y = \rho\sin\phi, z = z$

   Find the Jacobian of the transformation.

2. If u and v are functions of r and s also r and s are functions of x and y, prove that:

$$\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(x,y)}$$

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H. J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

UNIT 2 ORTHOGONAL CURVILINEAR CO-ORDINATES

CONTENTS

1.0 INTRODUCTION

In our elementary mathematics, we learnt about co-ordinate systems namely; (x, y, z), in the rectangular co-ordinates. In this unit, we will show that it is possible to work in other co-ordinate systems apart from the rectangular co-ordinate if we make the appropriate transformation. This is what we set to achieve in this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the orthogonal curvilinear co-ordinates
- determine the scale factors of transformation
- determine the elemental volume
- solve problems in other co-ordinate systems such as circular cylindrical and spherical co-ordinates.

3.0 MAIN CONTENT

3.1 Transformation of Co-Ordinates

Given the rectangular co-ordinates x, y, z, we can define a new co-ordinate system by the following equations expressible as:

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3) \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

Conversely, the relations as defined in (1) can be inverted to express

u_1, u_2, u_3 in terms of $x, y, z,$ whenever $x, y, z,$ and are suitably restricted.

Thus, at least in some region any point with the co-ordinates (x, y, z) has corresponding co-ordinates (u_1, u_2, u_3). We shall assume that the correspondence is unique.

Suppose a particle moves from point P in such a way that only u_1 is allowed to vary while u_2, u_3 are held constant, then it would generate a curve in space which is called $u_1 - curve$. Other curves u_2, and, u_3 are similarly generated.

3.1.1 Orthogonal Curvilinear Co-Ordinates

If one co-ordinate is held constant, we can determine successively three surfaces passing a point of space, these surfaces intersecting in the co-ordinate curves. When we chose a new co-ordinate in such a way that the co-ordinate curves are mutually perpendicular at each point, such co-ordinates are called Orthogonal Curvilinear co-ordinates.

3.1.2 The Scale Factors

Let $r = xi + yj + zk$ ……………. . . (2)

Represent the position vector of a point P in space. Then a tangent vector to the $u_1 - curve$ at P is given by:

$$U_1 = \frac{\partial r}{\partial u_1} = \frac{\partial r}{\partial s_1}\frac{ds_1}{du_1} \qquad \text{…………… ….} \quad (3)$$

Where s_1 arc length along the u_1 curve.

Since $\partial r / \partial s_1$ is a unit vector. We now write

$$U_1 = h_1 u_1 \qquad \text{………………} \ (4)$$

Where u_1, is the unit vector tangent to the u_1 curve in the direction of increasing arc length and $h_1 = ds_1/du_1$ is the length of U_1. If we consider the other co-ordinate curves similarly, we thus write

$$U_1 = h_1 u_{1,}, \ U_2 = h_2 u_2 \ , \ U_3 = h_3 u_3 \qquad \text{………………} \ (5)$$

Where $u_k \, (k = 1,2,3)$ is the unit vector tangent to the u_k curve, and

$$h_1 = \frac{ds_1}{du_1} = \left|\frac{\partial r}{\partial u_1}\right|, \quad h_2 = \frac{ds_2}{du_2} = \left|\frac{\partial r}{\partial u_2}\right|, \quad h_3 = \frac{ds_3}{du_3} = \left|\frac{\partial r}{\partial u_3}\right| \qquad \text{…………………} \ (6)$$

Putting these equations in the differential forms, we have the following expressions:

$$ds_1 = h_1 du_1, \quad ds_2 = h_2 du_2, \; ds_3 = h_3 du_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

h_1, h_2, and, h_3 are called the scale factors.

The co-ordinates curves are said to be orthogonal if:

$$U_1.U_2 = U_2.U_3 = U_3.U_1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

3.1.3 The Elemental Volume

The elemental volume is defined as

$$d\tau = h_1 h_2 h_3 du_1 du_2 du_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Example: The transformation from rectangular to cylindrical co-ordinates is defined by the transformations

$$x = \rho\cos\phi, y = \rho\sin\phi, z = z$$

(a) Prove that the system is orthogonal

(b) Find ds^2 and the scale factors

(c) Find the Jacobian of the transformation and the volume element.

Solution:

Let $e = (e_1, e_2, e_3)$ be a unit vector in the cylindrical co-ordinates

We have:

$$dr = \frac{\partial r}{\partial \rho} d\rho + \frac{\partial r}{\partial \phi} d\phi + \frac{\partial r}{\partial z} dz$$

$$= \; h_1 d\rho e_1 + h_2 d\phi e_2 + h_3 dz e_3$$

But

$$dr.dr. = \left(\left(\frac{\partial r}{\partial \rho} d\rho \right)^2 + \left(\frac{\partial r}{\partial \phi} d\phi \right)^2 + \left(\frac{\partial r}{\partial z} dz \right)^2 \right) + 2\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial \phi} d\rho d\phi + 2\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial z} d\rho dz + \; 2\frac{\partial r}{\partial \phi}.\frac{dr}{dz} d\phi dz$$

$$= \; h_1^2 (d\rho)^2 + h_2^2 (d\phi)^2 + h_3^2 (dz)^2 + 2h_1 h_2 d\rho d\phi e_1 e_2 + 2h_2 h_3 d\phi dz \; e_2.e_3 \; + 2h_1 h_3 d\phi dz e_1.e_3$$

Consider

$$r = \rho\cos\phi i + \rho\sin\phi j + zk$$

$$\frac{\partial r}{\partial \rho} = \cos\phi i + \sin\phi j$$

$$\frac{\partial r}{\partial \phi} = -\rho\sin\phi i + \rho\cos j$$

$$\frac{\partial r}{\partial z} = k$$

Now

$$\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial \phi} = -\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi = 0$$

Also,

$$\frac{\partial r}{\partial \rho}.\frac{\partial r}{\partial z} = \frac{\partial r}{\partial \phi}.\frac{\partial r}{\partial z} = 0$$

From (a) part we have:

$$dr.dr = ds^2 = h_1^2(d\rho)^2 + h_2^2(d\phi)^2 + h_3^2(dz)^2$$

$$h_1 = \left|\frac{\partial r}{\partial \rho}\right|, h_2 = \left|\frac{\partial r}{\partial \phi}\right|, h_3 = \left|\frac{\partial r}{\partial z}\right|$$

$$h_1 = 1, h_2 = \rho,\ h_3 = 1$$

$$ds^2 = (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2$$

(c) The Jacobian of the transformation is

$$= \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

Thus, the volume element dV is given as

$$dV = \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\right| d\rho d\phi dz = \rho d\rho d\phi dz$$

3.2 Gradient, Divergence, Curl and Laplacian in Orthogonal Curvilinear Co-Ordinates

If ϕ is a scalar function and

$$A = A_1 e_1 + A_2 e_2 + + A_3 e_3$$

is a vector function of orthogonal curvilinear co-ordinates u_1, u_2, u_3 then, we have the following results:

(1) Gradient: $\nabla \phi = \text{grad}\, \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1} e_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2} e_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3} e_3$

(2) Divergence of A: $\nabla . A = divA$ =

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right)$$

(3) Curl A = $\frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$

(4) $\nabla^2 \phi = \text{Laplacian of } \phi$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right)\right]$$

3.2.1 Special Orthogonal Co-Ordinate Systems

In this section, we shall examine some special orthogonal co-ordinate systems we usually come across in mathematics.

1. Cylindrical Co-ordinates (ρ, ϕ, z)

Here our transformation is the form:

$$x = \rho \cos \phi \quad y = \rho \sin \phi, z = z$$

Where $\rho \geq 0, \quad 0 \leq \phi < 2\pi, \quad -\infty < z < \infty$

$h_\rho = 1, \; h_\phi = \rho, \quad h_z = 1$

2. Spherical Co-ordinates (r, θ, ϕ)

Here the transformation is of the form:

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$, where $r \geq 0$,

$0 \leq \phi < 2\pi, \quad 0 \leq \phi \leq \pi$

$h_r = 1, h_\theta = r, h_\phi = r\sin\theta$

3. Parabolic Cylindrical Co-ordinates (u, v, z)

$x = \frac{1}{2}(u^2 - v^2), y = uv, z = z$, where $-\infty < u < \infty, \; v \geq 0$,

$-\infty < z < \infty, \quad h_u = h_v = \sqrt{u^2 + v^2}, \quad h_z = 1$ in cylindrical co-ordinates.

4. Paraboloidal Co-ordinates (u, v, ϕ).

Here, the transformations are given by:

$x = uv\cos\phi, y = uv\sin\phi, z = \frac{1}{2}(u^2 - v^2), u \geq 0, v \geq 0, 0 \leq \phi < 2\pi$

$h_u = h_v = \sqrt{u^2 + v^2}, h_\phi = uv$

Other special co-ordinates exist which include, elliptic cylindrical co-ordinates, prolate spheroidal co-ordinates, bipolar co-ordinates, ellipsoidal co-ordinates etc.

Consideration of the details of these co-ordinates will be left as exercise.

4.0 CONCLUSION

We have studied orthogonal co-ordinate systems in this unit; we have also identified some special co-ordinates systems that are orthogonal. Study this unit carefully before proceeding to the next unit of this course.

5.0 SUMMARY

Recall that if one co-ordinate is held constant, we can determine successively three surfaces passing a point of space, these surfaces intersecting in the co-ordinate curves. When we chose a new co-ordinate in such a way that the co-ordinate curves are mutually perpendicular at each point, such co-ordinates are called orthogonal curvilinear co-ordinates. We have also considered various types of these orthogonal systems particularly those for practical applications .You are to study them properly for better understanding.

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that a cylindrical co-ordinate system is orthogonal.
2. Express the velocity v and acceleration of a particle in cylindrical co-ordinates.
3. Find the square of the element of arc length in cylindrical co-ordinates and determine the corresponding scale factors.

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H. J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.