MODULE 4 COMPLEX VARIABLES

Unit 1 Complex Numbers
Unit 2 Polar Operations with Complex Numbers

UNIT 1 COMPLEX NUMBERS

CONTENTS

1.0 INTRODUCTION

The solution to the equation $x^2+1=0$ has no real roots because there is no real number whose square root is -1. In order to solve problem such as this, mathematicians evolve a way out of this logjam by assuming that there exist a number $i=\sqrt{-1}$. With this, we can conclude that the roots of the equation $x^2+1=0$ are $x=\pm i$. Similarly, we find that the roots of the equation $x^2-2x+5=0$ are $x=1\pm 2i$.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define complex numbers
- perform mathematical operations with complex numbers
- find modulus and argument of complex numbers
- solve exercises on complex numbers.

3.0 MAIN CONTENT

3.1 Definition of Complex Numbers

Given that a and b are real numbers, then the number $c=a+ib$ is called a complex number. a and b are known as the real and imaginary parts of the complex number respectively. When $a=0$ the complex number is purely imaginary and when $b=0$ then the complex number is real. The conjugate of the complex number c is denoted by:
$\bar{c}=a-ib$

SELF-ASSESSMENT EXERCISE

Find the conjugate of the following expressions:

i. 3-3i ii. 2i iii. -3+4i iv. 3-4i

3.2 Operations with Complex Numbers

In this section, we shall consider some mathematical operations on complex numbers.

(1) Note that in complex number,

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

(2) $(a+ib)-(c+id)=(a-c)+i(b-d)$

(3) $(a+ib)(a-ib)=a^2+b^2$ since $i^2=-1$

(4) If $a+ib=c+id$ then $a=c$ and $b=d$

(5) $\dfrac{a+ib}{c+id}=\dfrac{(a+ib)}{(c+id)}.\dfrac{(c-id)}{(c-id)}=\dfrac{(a+ib)(c-id)}{c^2+d^2}$

$$=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

SELF-ASSESSMENT EXERCISE

i. Find the real and imaginary parts of

$$z=\frac{(1+i)(2+i)}{(3-i)}$$

ii. Let $z_1=3-6i$ and find:

(a) z_1z_2 $(b)\dfrac{z_1}{z_2},(c)\dfrac{z_2}{z_1}$

iii. Simplify

(a) $(5-9i)-(2-6i)+(3-4i)$

(b) $(4+7i)(2+5i)$

iv Multiply $(4-3i)$ by an appropriate factor to give a product that is entirely real. What is the result?

3.3 Modulus and Argument of a Complex Number

Let r be the length of OP, suppose the <XOP $=\theta,$ then $r=\sqrt{x^2+y^2}$ and $\tan\theta=\frac{y}{x}$, r is called the modulus of z and written $|z|, \theta$ is called the argument or amplitude of and written as $\arg z$ or amz .

Examples:

1. Find the modulus and argument of the complex number

$$z=\frac{(1+i)(2+i)}{(3-i)}$$

Solution:

$$z=\frac{(1+i)(2+i)}{(3-i)}=\frac{2+2i+i-1}{3-i}=\frac{1+3i}{3-i}$$

Therefore,

$$z=\frac{1+3i}{3-i}\frac{(3+i)}{(3+i)}=\frac{(3+9i+i-3}{10}=\frac{10i}{10}$$

Hence, $z=i$ therefore $|z|=1$ and arg z=$\frac{\pi}{2}$

2. If $x+iy=a+\frac{b(1+it)}{(1-it)}$ where a and b are real constant and $x,y,t,$ are real variables show that the locus of the point (x,y) as t, varies as a circle.

Solution:

Let $x+iy=a+\frac{b(1+it)}{(1-it)}$

$$= a+\frac{b(1+it)}{(1-it)}.\frac{(1+it)}{(1+it)}$$

$$= a+\frac{b(1-t^2)}{1+t^2}+\frac{2bit}{1+t^2}$$

Equating the real parts and the imaginary parts in each side of the equation, we have:

$$x = \frac{b(1-t^2)}{1+t^2}, y = \frac{2bt}{1+t^2}$$

Thus,

$$(x-a)^2 + y^2 = b^2$$

Hence, the locus of the point (x, y) is a circle centre (a,0) and radius b.

We may represent complex numbers in the polar form as follows:

$$z = x + iy = r\cos\theta + ir\sin\theta$$

Compare coefficients then

$$x = r\cos\theta, y = r\sin\theta$$

We refer to this as the polar representation of the complex numbers.

4.0 CONCLUSION

We have shown the way to handle complex numbers. we shall deal with some problems into detail in complex variables.

5.0 SUMMARY

Recall that with clearly defined notation you can handle complex number as we handle real numbers ordinarily in algebra. You should study carefully before moving to the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Establish the following results:

(a) $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2), but, \mathrm{Re}(z_1 z_2) \neq \mathrm{Re}(z_1)\mathrm{Re}(z_2)$ in general

(b) $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1 \ \mathrm{Im}(z_2), but, \mathrm{Im}(z_1 z_2) \neq \mathrm{Im}(z_1)\mathrm{Im}(z_2)$ in general

(c) $|z_1 z_2| = |z_1||z_2|, but, |z_1 + z_2| \neq |z_1| + |z_2|,$ in general

2.. Express the following quantities in the form a + ib where a and b are real

(a) $(1+i)^3$ (b) $\frac{1+i}{1-i}$ (c) $\sin(\frac{\pi}{4} + 2i)$

3.. Prove the following

(a) $z + \bar{z} = 2\mathrm{Re}(z)$ (b) $z - \bar{z} = 2i\,\mathrm{Im}(z)$ (c) $\mathrm{Re}(z) \leq |z|$

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

Keisler, H. J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

UNIT 2 POLAR OPERATIONS WITH COMPLEX NUMBERS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall examine complex numbers in polar forms. The polar form of complex numbers presents interesting results which will be examined in this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

express complex numbers in polar form

- carry out multiplication and division of complex numbers
- recall the Demoivre's theorem and apply it appropriately
- find roots and work with fractional powers of complex numbers
- solve correctly the exercises that follows after the unit.

3.0 MAIN CONTENT

3.1 Multiplication and Division of Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then,

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

From the above, you could see that

$$|z_1 z_2| = |z_1|.|z_2|$$

We also note that

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$, and that

$$\frac{z_1}{z_2} = \frac{r_1(Cos\theta_1 + iSin\theta_1)}{r_2(Cos\theta_2 + iSin\theta_2)}$$

$$= \frac{r_1}{r_2}[(Cos\theta_1 + iSin\theta_1)(Cos\theta_2 - iSin\theta_2)]$$

$$= \frac{r_1}{r_2}[Cos(\theta_1 - \theta_2) + iSin(\theta_1 - \theta_2)]$$

Therefore,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg(\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$$

3.2 Demoivre's Theorem

Recall that:

$$(Cos\theta_1 + iSin\theta_1)(Cos\theta_2 + iSin\theta_2) = Cos(\theta_1 + \theta_2) + iSin(\theta_1 + \theta_2)$$

Note that,

$$(Cos\theta_1 + iSin\theta_1)(Cos\theta_1 + iSin\theta_1) = Cos2\theta_1 + iSin2\theta_1$$

This is equivalence to

$$(Cos\theta_1 + iSin\theta_1)^2 = Cos2\theta_1 + iSin2\theta_1$$

Also,

$$(Cos\theta_1 + iSin\theta_1)^3 = Cos3\theta_1 + iSin3\theta_1$$

If we continue in this way, we find that:

$$Cos\theta_1 + iSin\theta_1)^n = Cosn\theta_1 + iSinn\theta_1$$

This is known as the Demoivre's theorem for positive integer index.

It can be shown that the theorem is true for all rational values of n.

Now suppose n is a negative integer and we let n = -m where m is a positive integer then,

$$(Cos\theta + iSin\theta)^{-m} = \frac{1}{(Cos\theta + iSin\theta)^{m}}$$

$= \; Cos(-m\theta) + iSin(-m\theta) = Cos(n\theta) + iSin(n\theta)$

We can also prove for fractions. Recall that by Demoivre's theorem

$$(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta)^{q} \; = \; Cosp\theta + iSinp\theta = (Cos\theta + iSin\theta)^{p}$$

It follows that $Cos\frac{p}{q}\theta + iSin\frac{p}{q}\theta$ is a qth root of $(Cos\theta + iSin\theta)^{P}$

Demoivre's theorem has been proved for all rational values of n.

We need to find other values of $(Cos\theta + iSin\theta)^{\frac{p}{q}}$.

To do this, suppose that:

$$(Cos\theta + iSin\theta)^{\frac{p}{q}} = \rho(Cos\phi + iSin\phi)$$

Then,

$$(\cos\theta + i\sin\theta)^{p} = \rho^{q}(\cos\phi + i\sin\phi)^{q} \Rightarrow \cos p\theta + i\sin p\theta = \rho^{q}(\cos q\phi + i\sin q\theta)$$

Equating the real and imaginary parts, we have

$$\cos p\theta = \rho^{q}\cos q\phi ; \sin p\theta = \sin q\phi$$

By squaring and adding, we obtain

$\rho^{2q} = 1$ and since $\rho,$ the modulus of a complex number is +ve $\rho = 1$ therefore

$\cos\rho\phi = \cos q\phi ; \sin\rho\phi = \sin q\phi$, and these equation are satisfied by

$q\phi = p\theta + 2k\pi ; k = 0$ or any integer.

Therefore,

$$\phi = \frac{p\theta + 2k\pi}{q}$$

3.3 Roots and Fractional Power of a Complex Number

If n is a positive integer, the nth roots of a complex number are by definition the value of ω which satisfies the equation

$$\omega^n = z$$

If $\omega = \rho(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$ then

$$\rho^n(\cos^n\phi + i\sin^n\phi) = r(\cos\theta + i\sin\theta) \quad \text{where}$$

$\rho^n = r$ and $n\phi = \theta + 2k\pi$ k is an integer or zero. By definition ρ, and, r are +ve, such that $\rho = \sqrt[n]{r}$ also, $\phi = \dfrac{\theta + 2k\pi}{n}$

Taking in succession the values of k = 0, 1, 2, 3… n, we find that

$\dfrac{\cos\theta + 2k\pi}{n} + i\dfrac{\sin\theta + 2k\pi}{n}$ has n distinct values. Hence, there are n distinct nth roots of z given by the formula.

$$\omega_k = \sqrt[n]{r}[\frac{\cos\theta + 2\pi k}{n} + i\frac{\sin\theta + 2\pi k}{n}], \quad \text{k=0, 1, 2, 3,…, n-1}$$

In a situation where n is a rational number say $n = \dfrac{p}{q}, p, and, q$ are integers and q is +ve, the value of z^n are the values of ω which satisfy the equation

$$\omega^q = z^p$$

Hence if $z = r(\cos\theta + i\sin\theta)$ then the q values of $z^{\frac{p}{q}}$ given by the formula

$$\omega_m = \sqrt[q]{r^p}\left[\frac{\cos\theta + 2m\pi}{q} + i\frac{\sin\theta + 2m\pi}{q}\right], \text{ where}$$

$\sqrt[q]{r^p}$ is the unique positive qth root of r^p

Example:

Find the fifth roots of -1

Solution:

Recall that:

-1= $\cos\pi + i\sin\pi$

Now if

$z^5 = -1 = [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]$, k=0,1,2,3,…,

Therefore,

$$z = \frac{\cos(\pi + 2k\pi)}{5} + i\frac{\sin(\pi + 2k\pi)}{5}$$

$k = 0,1,2,3,4,$ hence, the solutions are:

$$z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$

$$z = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$

$$z = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}$$

$$z = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$

$$z = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

3.4 The nth Roots of Unity

We recall that $\cos 0 + i\sin 0 = 1$ this implies that:

$1 = \cos 2\pi k + i\sin 2\pi k,$ k=0, 1, 2, 3…

If ω denotes the root $\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, k=0, 1, 2, 3…, then nth root of unity may be written in the form

$1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$

We see that they form a geometric progression whose sum $\frac{1-\omega^n}{1-\omega}$ is equal to 0.

We also note that the nth root of unity is represented in the Argand diagram by points which are vertices of a regular polygon of n sides inscribed in the circle.

Example:

Solve the equation $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ and deduce that

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

Solution:

We know that:

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = \dfrac{z^7 - 1}{z - 1}$, hence we consider the equation

$$z^7 - 1 = 0$$

We also note that:

$1 = \cos 0 + i \sin 0 = \cos 2\pi k + i \sin 2\pi k$, hence

$$z = \frac{\cos 2\pi k}{7} + i \sin\frac{2\pi k}{7}, k = 0,1,2,3,4,5,6$$

Equation $z^7 - 1 = 0$ is satisfied by

$z = 1, and, by, z = \dfrac{\cos 2\pi k}{7} + i\dfrac{\sin 2\pi k}{7}$, therefore the given equation is satisfied by

$$z = \cos\pm\frac{2\pi k}{7} + i \sin\pm\frac{2\pi k}{7}, k = 1,2,3,4,.5,....$$

The sum of these roots is 226

$$2\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right]$$

But from the given equation the sum of the roots is also -1.

Therefore,

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

4.0 CONCLUSION

In this unit, we have studied some theorems and determined the roots of equation using complex variables. You are required to study this unit properly before attempting to answer questions under the Tutor-Marked Assignment.

5.0 SUMMARY

You recall that you learnt about Demoivre's theorem, both for integer quantity and fractional quantity. Also, you learnt about roots of unity among others. You are to study them properly in order to be well equipped for the next course in mathematical methods.

6.0 TUTOR-MARKED ASSIGNMENT

1. Obtain the roots of the equation
   $3z^2 - (2+11i)z + 3 - 5i = 0$ in the form a+ib where a and b are real.
2. Express $\cos^3\theta\sin^4\theta$ as a sum of cosines of multiple of θ
3. Prove that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

   By putting

   $x = \cos^2\theta$ or otherwise, show that the roots of the equation

   $64x^3 - 96x^2 + 36x - 3 = 0$ are $\cos^2\left(\frac{\pi}{18}\right), \cos^2\left(\frac{5\pi}{18}\right), \cos^2\left(\frac{7\pi}{18}\right)$ and deduce that

   $$\sec^2\left(\frac{\pi}{18}\right) + \sec^2\left(\frac{5\pi}{18}\right) + \sec^2\left(\frac{7\pi}{18}\right) = 12$$

7.0 REFERENCES/FURTHER READING

Wrede, R. C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

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