MODULE 2

Unit1 ElementaryPrincipleOf TheTheoryOf Probability
Unit 2 Bayes Theorem
Unit3 Discrete Random Variables
Unit 4 Geometric Random Variable
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UNIT1 ELEMENTARYPRINCIPLEOF THETHEORYOF PROBABILITY

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1.0 INTRODUCTION

In module1, we considered some elementary methods of mathematics of counting essentialfor determiningprobabilitiesof events.Inthis module,wecontinueour study by discussinghowtoapplytheknowledgegainedin module1 todetermineprobabilitiesof eventsandgeneralpropertiesof probabilities.Wepresentsometheoremsand definitions thatarebasictotheunderstandingofcommonlyencounteredproblems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use the principle of theory of probability to solve related probability questions
- state the properties of probability
- define mutually exclusive events and complementary event
- find the probability of two mutually exclusive events
- solve probability of an event A given that an event B has occurred

3.0 MAIN CONTENT

3.1 ElementaryPrincipleoftheTheoryofProbability

In any random experiment there is always uncertainty as to whether a particular event will or will not occur. As a measure of the chance or probability with which we can expect the event to occur it is convenient to assign a number between 0 and 1. If we are sure or certain that the event will occur we say that its probability is 100% or 1, but if we are sure that the event will not occur we say that its probability is zero.

Example2.1

Supposeafairdieisrolled once.Thereare sixpossibleoutcomes.Thesamplespaceis

1,2,3,4,5,6.

letA_1,A_2andA_3,representthefollowingevents

A_1= "anevennumberoccurs"

A_2= "anodd numberoccurs

A_3=aprimenumber occurs"

ThesixpossibleoutcomesinΩareequallylikely.Theprobabilitythat Aoccursis denotedbyP(A)andtheprobabilitythatAdoesnotoccurbyP(A^C).ifweletn$_A$ bethenumberofoutcomesthathaveattributeA,then

A_1 {2,4,6}

A_2 {1,3,5}

A_3 {2,3,5}

Thus,

$$P(A_1) = \frac{nA_1}{n\Omega} = 3/6 = 1/2,\ p(A_2) = \frac{3}{6},\ P(A_3) = \frac{3}{6}$$

Example2.2

Supposethataboxcontains10itemsofwhicharedefective.Twoitemsareselectedat randomwithoutreplacement.Find theprobabilitiesthat:

(i) bothitemsarenon-defective (ii) only one item isdefective (iii) bothitemsaredefective: (iv) atleastoneitem isdefective

LetA_1,A_2,A_3,A_4,denotetheevents

"bothitemsarenon-defective

"onlyone item isdefective"

"bothitemsaredefective"

"atleastoneitem isdefective"

respectively.Thenumberofwaysof selecting2itemsfrom 10is $^{10}C_2$ 45*ways*.

Sothereare45 elementsinthesamplespace.

(i) Thenumberof waysofselecting2itemsfrom thenon-defectiveitemsis $^6C_2$= 15.

ThatisA$_1$canoccurin15ways.Thus.

$$P(A_1) = \frac{^6C_2}{^{10}C_2} = \frac{nA_1}{n\Omega} = \frac{15}{45} = 1/3$$

(ii) Thenumberof waysofselecting1item from the4defectiveitemsand 1item fromthe6non-defectiveitemsis $^4C_1 \times {}^6C_1$=24so,A$_2$canoccurin2ways.

Thus,

$$P(A_2) = \frac{^4C_1 \times {}^6C_1}{^{10}C_2} = \frac{24}{45} = \frac{8}{15}$$

Similarly

(iii) $P(A_3) = \frac{^4C_2}{^{10}C_2} = \frac{6}{45} = \frac{2}{15}$

(iv) Atleastonedefectivemeans1 or2defectiveitems.So,

P(A4)= P(1defectiveitem)+ P(2defectiveitems)=P(A2)+ P(A3)$= \frac{8}{15} + \frac{2}{15} = 2/3$.

Example2.3

Supposeafairdieisrolledtwice.Findtheprobabilitythatthesum ofthenumbers onthe twofacesis(i) even,(ii)lessthan5.
ThesamplespaceΩ,consistsof36elements.

(i) LetAbetheevent "thesumofthetwo facesiseven".Possibleoutcomeare:
(1,1) (1,3) (1,5) (2,2) (2,4) (2,6),…
(3,1) (5,1) (4,2) (6,2),…

A canoccurin 18ways. Thus,

$$P(A) = \frac{18}{36} = 1/2$$

(ii) LetBbe theevent “thesumisless than5”. Boccursifthesumis2,3,or4.The sum is2iftheoutcomeis(1,1) thesumis3iftheoutcomeis(1,2) or(2,1) andthe sum isfour iftheoutcomeis(3,1),(1,3) or(2,2).ThereforeBcanoccurin

1+2+3=6ways

Thus

$$P(B) = \frac{6}{36} = {}^{1}/_{6}.$$

Example2.4

3ballsaredrawnatrandomwith replacementfromaboxcontaining8redand3white balls. Findtheprobability that(i)all3 arered;(ii)1isredand 2 arewhite.

Thesamplespaceconsistsof11^3possibleoutcomes.LetA be“theevetall3 arered”and

B theevent “1isredand2white”

$$P(A) = \frac{nA}{n\Omega} = \frac{8^3}{11^3}$$

(Thefirstredcan bechosenin8waysthesecondin 8waysand thethirdRedin8ways).

$P(B)=\frac{nB}{n\Omega}$ nB=8,3.3+3.8.3+3.3.8
,

(8×3×3) =numberof ways of pickingthefirsttobered,secondwhiteandthirdwhite(RWW),3×8× 3isfor WRWand3×3×8isfor WWR).Thus

$$P(B) = \frac{3.8.3.3}{11^3}.$$

3.2Propertiesof Probability

Definition2.1:Mutually Exclusiveevents

TwoeventsA$_1$andA$_2$aremutuallyexclusiveif anonly if

$A_1 \cap A_j = \phi$ foreveryi≠j.

Thatis,twoormoreeventsaremutuallyexclusiveif notwoof themhavepointsin common.Inexample2.1
$A_1=\{2, 4, 6\}$
$A_2=\{1, 3, 5\}$

$A_1 \cap A_2 = \phi$,

Therefore A_1 and A_2 are mutually exclusive events.

Definition 2.1

If A_1 and A_2 are any two events, $A_1 \cap A_2$ is the event that occurs if and only if both A_1 and A_2 occur. In general, if $A_1, A_2, \ldots, A_n$ occur.

In example 2.1. $A_1 \cap A_2 \cap A_3$ is the event that an even prime occurs.

$A_1 \cap A_3 = \{2\}$

Thus

$P(A_1 \cap A_3) = \frac{2}{6}$

The event A_2 A_3 is the event that an odd prime occurs $A_2 \cap A_3 = \{3,5\}$. Thus

$P(A_1 \cap A_2) = \frac{2}{6} = \frac{1}{3}$.

Complementary Event

The event "A occurs" and "A does not occur" are mutually exclusive. The event "A does nt occur" is called the complement of A and is denoted by A^C.

Theorem 2.1

If A^C is the complement of an event A, then $P(A^c) = 1 - P(A)$. This theorem states that the probability that an event will not occur is equal to 1 minus the probability that it will occur.

In example 2.2(iv); The event "no defective item" is the same as the event both items are non-defective".

From Theorem 2.1, we have

$P(A_4) = 1 - P(A_4^c)\, P(A_1)\, 1 - \quad = 1 - \frac{1}{3} = \frac{2}{3}$

$A_4^c = A_1$ since the event "no defective item" is the same as the event" both items are non-defective.

The following events are complementary events

(i) atmost2 andgreaterthan2 (ii) atleast4 andless than4.

Definition2.2

If A_1andA_2areany twoevents,$A_1 \cup A_2$istheeventthatoccurs iftleastoneof A_1andA_2occurs.

Ingeneral,if$A_1, A_2, \ldots, A_n$are any events,$A_1 \cup A_2$istheeventthatoccurs if atleastone of A_1i= 1,2,…,n,occurs.

In example2.1,theevent$A_1 \cup A_3$istheeventthatanevenoraprimenumber occurs.

$A_1 \cup A_3=\{2, 3,4,5,6\}$.

Thus

$P(A_1 \cup A_2)= \frac{n(A_1 \cup A_2)}{n\Omega} \quad \frac{5}{6}$

Theevent$A_1 \cup A_2$isthe eventthataneventoranoddnumber occurs.

$A_1 \cup A_2=\{1,2,3,4,5,6\} = \Omega$,

Thus

$P(A_1 \cup A_2)= P(\Omega)= = 1. \quad \frac{6}{6}$

Theorem2.2

If A_1andA_2areany events,then

$P(A_1 \cup A_2)= P(A_1)+P(A_2)–P(A_1 \cap A_2)$.

Theproof ofthistheoremfollowsdirectlyfromthedefinition

$$P(A_1 \cup A_2) = \frac{n(A_1 \cup A_2)}{n\Omega}$$

$$n(A_1 \cup A_2) = nA_1 + nA_2 - n(A_1 \cap A_2)$$

dividingthrough bynΩtheresultfollows.

Corollary

(i) If A_1 and A_2 are mutually exclusive event then

$P(A_1 \cup A_2) = P(A_1) + P(A_2)$

since A_1 and A_2 are mutually exclusive,
$A_1 \cap A_2 = \phi$

$P(A_1 \cup A_2) = 0$

hence

(ii) If A_1 and A_2 are any two events

$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

In Example 2.1 above,

$P(A_1 \cup A_3) = P(A_1) + P(A_3) - P(A_1 \cap A_3) = \quad = .$ $\frac{1}{2} + \frac{1}{2} \quad \frac{1}{6} \quad \frac{5}{6}$

(iii) If A and B are any two events defined on the sample space then
$P(A) = P(A \cap B) + P(A \cap B^c)$

$P(B) = P(A \cap B) + P(A^c \cap B)$
In general, if $A_1, A_2, \ldots A_n$ are any n mutually exclusive events, then

$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + \ldots + P(A_n)$.

Theorem 2.2 can be generalized for n > 2 events. It can easily be shown that for any 3 events A_1, A_2, A_3

$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2)$

$- \; P(A_1 \cap A_3) – P(A_2 \cap A_3) + \ldots + P(A_3)$.

To see this, let $B = A_1 \cup A_2$, then

$P(A_1 \cup A_2 \cup A_3) = P(B \cup A_3) = P(B) + P(A_3) – P(B \cup A_3)$

$P(B) = P(A_1) + P(A_1) + P(A_2) – P(A_1 \cap A_2)$

$P(B \cap A_3)=P(A_3 \cap)= P\{(A_3 \cap(A_1 \cup A_2)\}= P\{(A_3 \cup A_1) \cup (A_3 \cap A_2)\}$

$=P(A_3 \cap A_1)+P(A_3 \cap A_1)+P(A_3 \cap A_2)- P(A_1 \cap A_2 \cap A_3)$

(bytheorem2.2).thus

$P(A_1 \cup A_2 \cup A_3)=P(A_1)+P(A_2)-P(A_1 \cap A_2)+ P(A_3)$

$- \; P(A_1 \cap A_3)-P(A_2 \cap A_3)+ P(A_1 \cap A_2 \cap A_3)$

Note:

$A_1 \cup A_2 \cup A_3$ istheeventthatatleastoneofthemwilloccur.

Definition2.3

Twoormoreeventsaresaidtobeexhaustiveiftheirunionequalsthewholesample space.In otherwords,$A_1,A_2,\ldots,A_n$areexhaustiveeventsif
$P(A_1 \cup A_2 \cup \cdots \cup A_n)=1$
Inotherwords itiscertainthatatleastoneof themwilloccur.

Note:

1. $0 \leq P(A) \leq 1$

2. $P(\Omega) =1$.

3. For ansequenceofmutuallyexcusiveevents$A_1,A_2,\ldots$

$$P(\cup_{i-1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Examples:

2.5 A box contains6 ballsnumbered1to6.Aballwas drawnfromtheboxatrandom.

Findtheprobability that thenumberontheballdrawnwaseither 1,2,or 6.

Solution

LetA_1,A_2,A_3 denotetheeventsthattheballdrawnwas1,2and6respectively. $A_1 \cup A_2 \cup A_3$ denotetheeventthenumberontheballdrawnwaseither1,2or6.
A_1,A_2andA_3aremutuallyexclusiveevents.Thus

$P(A_1 \cup A_2 \cup A_3)=P(A_1)+P(A_2)+ P(A_3)$

=1/6+1/6+1/6 =12.

2.6 Fourfairdicearetossedonce,whatistheprobabilitythatthesumofthenumbers onthefourdiceis23?

Solution

Thepossibleoutcomestoget23are

(6,6,6,5), (5,6,6,6),(6,5,6,6),(6,6,5,6). ThesamplespaceΩconsistsof 6^4elements.

Therefore,theprobabilitythatthesumofthenumbersonthefourdiceis23is$4/6^4$.

2.7 If $P(A_1)=2/3$,$P(A_1 \cap A_2)=¼$ and$P(A_1 \cup A_2)=5/6$,find$P(A_2)$.
Solution

Fromthe additionlawofprobability

$P(A_1 \cup A_2)=P(A_1)+P(A_2)–P(A_1 n A_2)$

Wehave

$P(A_2)= P(A_1 \cup A_2)$ $P(A_1 n A_2)–P(A_1)$=5/6+¼ - 2/3= 5/12.

2.8 Supposeafaironistossedthreetimes,whatistheprobabilitythatatleastone headoccurs?

Solution

LetA_1betheeventthatthefirsttosslandsheads,A_2theeventthatthesecondtoss lands,andA_3theeventthatthethirdtosslandsheads.
$A_1 \cup A_2 \cup A_3$istheeventthatatleastoneheadoccurs.

WearerequiredtocomputeP$(A_1 \cup A_2 \cup A_3)$ThecomplementofA$_1 \cup A_2 \cup A_3$is

$A_1^c n A_2^c n A^c$.Thatis,thefirstdoes notlandheads,the seconddoesnotandthe thirddoesnot.

$A_1^c n A_2^c n A_3^c$={T,T,T}.

Thus

$P(A_1^c \cap A_2^c \cap A_3^c) = 1/8$.

Hence

$P(A_1 \cup A_2 \cup A_3) = 1 - 1/8 = 7/8$.

2.9 If $A_1 \subset A_2$, then $P(A_1) \leq P(A_2)$

$A_2 = A_1 \cup (A_2/A_1)$.

Since A_1 and A_2/A_1 are mutually exclusive, we have
$P(A_2) = P(A_1) + P(A_2/A_1) \geq P(A_1)$
Since $P(A_2/A_1) \geq 0$.

2.9 If $A_1, A_2, \ldots, A_n$ are n events, then

$P(\bigcup_{i=1}^{n} A_1)^c =$ $P(\bigcup_{i-1}^{n} A_1{}^c$

From de Morgan's law, we have

$(\bigcup_{i=1}^{n} A_1)^c = \bigcap_{i=1}^{n} A_1^c$

Hence

$P(\bigcup_{i=1}^{n} A_1)^c =$ $P(\bigcup_{i=1}^{n} A_1{}^c)$

2.2 ConditionalProbability

TheconditionalprobabilityofaneventAgiventhataneventBhasoccurredisdenoted asP(A|B).theword"given"isrepresentedbytheuprightstroke.Wewishtodeterminethe probabilitythataneventA willoccur"conditionalon" theknowledgethatanother eventBhasoccurred.

Supposeafairdieisrolledanditisknownthatanevennumberappeareduppermost.Let A be the eventthat thenumberwasgreaterthan3andBtheeventthatthenumberthat appearedwaseven.Theproblemisto findtheconditionalprobabilitythattheeventA occurredgiventhattheeventB has occurred,P(A|B). sincewe knowthatthe numberwas even, the number must be either 2,4 or 6. Therefore, theconditional sample space contains3 elements.TheeventAoccursifthenumbershowingis4or6,thusP(A|B)=2/3.

WecanthereforedefineconditionalprobabilityofAgivenBasthenumberofwaysAnB

canoccurdividedbynumberofelementsin theconditionalsamplespace.Thatis, $P(A|B)$, $= \frac{n(A \cap B}{}$

$n\Omega_B$

Where $n\Omega_B \neq 0$, where Ω_B is the condtional sample space given that B has occurred.

Divide the numerator and denominator by $n\Omega$

$$P(A|B) = \frac{\frac{n(A\cap B)}{n\Omega_B}}{\frac{nB}{n\Omega_B}} = \frac{P(A\cap B)}{P(B)}.$$

In the above example,

n(A∩B)=2
n(B)=3

Definition 2.4

Let A and B be two events such that P(B)>0. Then the conditional probability of A given B, denoted by P(A|B) is defined to be $P(A|B) = \frac{P(A\cap B)}{P(B)}$ 2.2

Examples

2.11 Suppose a box contains 4 red balls and 3 black balls. Compute the probability that

(i) the second ball drawn is red if the first ball drawn was red; without replacement,
(ii) the second ball drawn is red if the first ball drawn was black

Solution

If the first ball drawn is red, there remains 6 balls, 3 red balls and 3 black balls.

The probability of the second ball being red is $\frac{3}{6}$ $\frac{1}{2}$. But if the first ball is black, the box is left with 4 red and 2 black so the probability of the second ball being red is then 4/6=2/3. Thus

P(2^{nd} is re/first was red)= 3/6/=½

P(2^{nd} is red/first was black)=4/6=2/3.

This shows that the probability of the event "the second ball drawn is red" depends on the colour of the first ball drawn.

2.12 Suppose two fair dice are rolled. If the sum of the numbers appearing is 6, what is the probability that one of the number is 2?

Solution

Let A be the event "one of the numbers is and B the sum is 6. there are five ways fortheeventBtooccur:(3,3),(2,4),(4,2),(5,1)and(1,5)andtherearetwoways for theeventAnBtooccur:(2,4) and(4,2).

Thus,
$P(A\ B)$ 2/36,$P(B)$ 5/36

Hence,
$P(AB) = \frac{P(A\ B)}{P(B)} 2/5.$

2.13 Therearetwochildreninafamily.Ifthereisatleastagirlinthisfamily,whatis theconditionalprobabilitythatbotharegirls.

Solution

Thesamplespaceis
$\{BB,GB,BG,GG\}$
LetAbetheevent "bothchildrenaregirlsandB"aleastagirlin thefamily.

B={GB, BG,GG},A={GG},A∩B={GG}.

$$P(A\ B) = \frac{P(AnB)}{P(B)}\ \frac{1/4}{3/4}\ \frac{1}{3}$$

2.14 Therearethreechildreninafamily.Ifthereisatleastoneboyandatmosttwo boys in thisfamily.Whatistheconditionalprobabilitythatthereareexactlytwo boys inthisfamily.

Thesamplespaceis
Ω={BBB,BBG,BGG,BGB,GBB,GBG,GGB,GGG}
LetBbetheevent"atleastoneboyandatmost2boysinthefamily"andletAbe theevent"exactly twoboys inthefamily".Then

B={BBG,BGG,BGB,GBB,GBG.GGB}
A∩B={BBG,BGB,GBB}.
Therefore

$$P(A\ B) = \frac{P(AnB)}{P(B)}\ \frac{3/8}{6/8}\ \frac{3}{6}\ \frac{1}{2}.$$

SELF-ASSESSMET EXERCISES1

i. A fairdieisthrowntwice.
   (i) Ifitisknownthatthesumofthenumbersappearingwas8,whatisthe probabilitythatthedifferencebetweenthetwonumberswas2.
   (ii) Ifitisknownthatthedifference thetwo numberswas3,what isthe

probabilitythatthesum ofthetwo numberswas 7?

ii. Twounbiaseddicearethrown once.Whatistheprobabilitythat
(i) atleastone5isthrown. (ii) thesumis10
(iii) thesumis10 giventhatno5isthrown?

4.0 CONCLUSION

5.0 SUMMARY

In this unit, the following were treated:

- Elementary principle of the theory of probability
- properties of probability
- conditional probability

6.0 TUTOR-MARKED ASSIGNMENT

1. SupposeeventsAandBaresunthatP(A)= 1/5, P(AnB)=1/6.
   Find(i)P(BIA); (ii)P(A^c B^c).
2. IfAandBaretwoeventsdefinedonthesameprobabilityspace,showthat:(i) P(A)= $P(A\cap B)+P(A\cap B^c)=P(B)+P(A\cap B^c)-P(A^c\cap B)$.
3. ProvethatP($A^c\cap B^c$)=1–P(A)–P(B)+ P(A ∩ B).
4. Supposeawellbalancedcoinistossedtwice. Findtheconditionalprobabilitythat
   (i) bothcoinsshowatailgiventhatthefirstshows ahead;
   (ii) bothareheadsgiventhatatleastoneofthem isahead.
5. A reddieandagreendiearerolledonce.Findtheconditionalprobabilitythat:
   (i) thenumberonred dieisodd,giventhatthesumofthetwonumbers showing is9;
   (ii) thesumofthetwonumbersis9giventhatoneofthenumbersisoddand theothereven?

7.0 REFERENCE/FURTHER READING

Harry Frank & Steven C. Althoen (1995). Statistics: Concepts and Applications. Cambridge University Press.

UNIT 2 BAYES THEOREM

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1.0 INTRODUCTION

Bayes' Theorem is often referred to as a theorem on the probability of causes because it enables one to find the probabilities of the various events A_1, A_2, …., A_n which can cause A to occur. Also, the notion of independence is a basic tool of probabilitytheory.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Use Bayes' theorem to calculate the probability of an event given the occurrence of other events that forms the sub-space of the sample space
- Calculation of the probabilities of independent events

3.0 MAIN CONTENT

3.1BayesTheorem

Supposeaboxcontainsrredballsandbblackballs.Twoballsaredrawnatrandomwithoutreplacement.Assumethattheprobabilityofdrawinganyparticularballis $\frac{1}{r+b}$

LetA_1 betheevent"thefirstballdrawnisredandletA_2 betheevent"thesecondballdrawnisred. Then

$$P(A_1)\ \frac{r}{r\ \ b}, P(A_2\ \ A_1)\ \frac{r\ \ 1}{r\ \ b\ \ 1}$$

$$P\ A_2\ \ A_1{}^c\ \ \frac{r}{r\ \ b\ \ 1}$$

Theprobability of the eventA_2dependson$A_1$and$A_1{}^c$. Thatis$A_2$is equivalenttoA_2n$A_1 \cap A_1{}^c$

$A_2 = A_1 \cap A_2$ or $A_1^c \cap A_2$.

Therefore,

$P(A_2) = P(A_2 \cap A_1) + P(A_2 \cap A_1^c) = P(A_1)P(A_1)P(A_2 \quad A_1) + P(A_1^c)P(A_2 \quad A_1^c)$.
Theorem2.3

IftheprobabilityofaneventB,dependsonkmutuallyexclusiveandexhaustiveevents

$A_1, A_2 + \ldots + A_k$,then

$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_k)$

$$= \sum_{i\ 1}^{k} P(B \quad A_1) \ \sum_{i\ 1}^{k} P(A_1) P(B\, A_1).$$

Example2.15

Supposeaboxcontains3redballs,2blackballsand5greenballs.Twoballsaredrawn atrandom withoutreplacement.Findtheprobabilitythatthesecondballdrawnisred.Let A_1 betheevent“thefirstballdrawnisredA_2 theeventthefirstballdrawnisblackandA_3,theevent“thefirstballdrawnisgreen.LetBbethee vent“thesecondballdrawnis red.TheeventBoccursif

(i) thefirst ball isredand thesecondisredor
(ii) thefirst ball isblackandthesecondisredor
(iii) thefirst ball isgreenandthesecondisred. Thus

$B = B \cap A_1$ or $B \cap A_2$ or $B \cap A_3$, $P(B \quad A_1) = 2/9$, $P(B\ A_2) = 3/9$, $P(B\ A_3) = 3/9$.

P(B)dependsonA_1, A_2, A_3whicharemutuallyand exhaustiveevents.Therefore,
$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)$

$= P(A_1)P(B\ A_1) + P(A_2)P(B\ A_2) + P(A_3)P(B\ A_3) = 3/10.2/9 + 2/10.3/9 + 5/10.3/9$
since$P(A_1) = 3/10$, $P(A_2) = 2/10$, $P(A_3) = 5/10$.

Thus

$P(B) = 1/90(6 + 6 + 15) = 27/90 = 3/10$.

Example2.16

SupposeafactoryhasthreemachinesM_1,M_2,M_3 whichproduce60%,30%and10%of thetotalproductionrespectively.Oftheiroutput,machineM_1 produces2%defective items,machineM_2produce3% defectiveitemswhilemachineM_3produces 4% defective items. Find theprobabilitythatapartselectedatrandomis defective.

Solution

LetBbe the event "apart selectedatrandom isdefective".A defectiveitemcouldhave beenproducedbyeithermachineM_1,M_2orM_3.Thus

$B=(B\cap M_1)U(B\cap M_2)U\ (B\cap M_3)$

Since$(B\cap M_1)$,$(B\cap M_2)$,$(B\cap M_3)$are mutuallyexclusive events.Thefollowing informationis containeddintheequation.

$P(M_1)=60\%=0.6, P(M_2)= 0.3, P(M_3)= 0.1$

$P(B|M_1)=0.02, P(B|M_2)= 0.03, P(B|M_3)= 0.04$

Hence

$P(B)=P(M_1)P(B|M_1)+P(M_2)P(B|M_2)+P(B|M_3)P(M_3)$

$=(0.6\times0.02) +(0.3\times0.03) +(0.1\times 0.04)= 0.002+0.009+0.004=0.025.$

supposeyouarenowasked,whatistheprobabilitythatagivendefectivepartwas producedby machineM_1.thatis,youaretofind$P(M_1|B)$=P(apartwasproducedby machineM_1giventhatthepartwasdefective).

NOTE:

1. $P(M_1|B)\neq P(B|M_1)$,but
2.$P(M_1nB)=P(BnM_1)$

$P(B)=\sum_{i1}^{3} P(M_1 nB)$

hence,

$$P(M_1|B) = \frac{P(M_i nB)}{\sum_{i1}^{3} P(M_i nB)} \quad \frac{P(M_i)P(B|M_i)}{\sum_{i1}^{3} P(M_i)P(B|M_i)} \qquad 2.4$$

Thus,

$$P(M_1|B) = \frac{0.6 \quad 0.02}{0.025} \quad \frac{0.012}{0.025} \quad 0.48$$

$$P(M_2|B) = \frac{0.3 \quad 0.03}{0.025} \quad \frac{0.009}{0.025} \quad 0.36$$

$$P(M_3|B) = \frac{0.1 \quad 0.04}{0.025} \quad \frac{0.004}{0.025} \quad 0.16$$

Equation(2.4)isanexampleof Bayestheoremwhichmaybestatedasfollows.

BayesTheorem

If$A_1,A_2,\ldots A_k$aresetofmutualexclusiveandexhaustiveeventsinasamplespaceΩand

B isany othereventinΩsuchthatP(B)>0,then

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{k} P(A_i)P(B|A_i)}; i \quad 1,2,\ldots,k \qquad 2.5$$

Example2.17

Supposeacollegeiscomposedof70%maleand30%femalestudents.Itisknownthat

40%ofthemalestudentsand20%ofthefemalestudentssmokecigarette.Findthe probabilitythatastudentobserved smokingacigaretteismale?

LetM,FdenotemaleandfemalerespectivelyandSdenotessmoker.Theaboveproblem containsthefollowinginformation.
P(M) =P(A studentselectedatrandomismale)=0.7

P(F) =P(A studentselectedatrandomisfemale)=0.3

P(S|M) =P(astudentselectedatrandomsmokes giventhattheselectedstudentismale)=0.4

P(S|F)=P(astudentselectedatrandomsmokesgiventhattheselectedstudentisfemale)=0.2

ByBayestheorem,wehave

P(M|S) =P(astudent observed smoking ismale)

=P(Astudentselectedatrandomismalegiventhattheselectedstudentisa smoker)

$= \frac{P(MnS)}{P(S)}$

WhereP(S) =P(astudentselectedatrandomisasmoker)

=P(S∩M)+ P(S∩F)=P(M)P(S|F)

=(0.7×0.4)+ (0.3×0.2) =0.28+0.06=0.34

Thus

$$P(M|S) = \frac{P(M)P(S \mid M)}{P(S)} \quad \frac{0.7 \quad 0.4}{0.34} \quad \frac{0.28}{0.34} \quad \frac{14}{17}.$$

Example2.18

A tablehasdrawers.Drawer1 containstwo redand fiveblackbiros, drawerII contains fourredandthreeblackbirosanddrawerIIIcontainsoneredandsixblackbiros.A drawerischosenatrandomandabiroischosenfromthedrawer.Findtheprobability that

(i) thebiro chosenisblack
(ii) thebiro chosenisfrom drawerI ifthechosenbiro isblack.

Solution

Let P(i)denotetheprobabilitythatdrawer Iisselected (I–1,2,3)andRandBrepresentingredandblackbirosrespectively.

Then

P(1) =1/3,P(2) =1/3,P(3),=1/3

P(B|1)= 5/7,P(B|2)=3/7,p(B|3)=6/7

Fromequation(2.3)

(i) P(B)=P(Bn1)+P(Bn2)+P(Bn3)= P(1)P(B|1)+P(2)P(B|2)+ P(3)P(B|3)

=1/3.5/7+1/3.3/7+1/3.6/7=1/3(5/7+3/7+6/7)= 2/3.

(ii) P(1|B)=P(thebirochosencamefromdrawer1giventhatthechosenbirois black).

$$= \frac{P(1 \cap B)}{P(B)} = \frac{P(B \cap 1)}{P(B)} = \frac{P(1)P(B|1)}{P(B)} = \frac{\frac{1}{3} \cdot \frac{5}{7}}{\frac{2}{3}} = \frac{5}{14}$$

Definition2.5

LetA,BandCbethreeeventssuchthatP(C)>0.thentheconditionalprobabilityofA BgivenCis definedby

P(A B|C)=P(A|C)+P(B|C)–P(A∩B|C).

SELF-ASSESSMENT EXERCISE2

1. Supposethataboxcontains5ballslabeled1to5.twoballsaredrawnatrandom(oneafter theotherwithoutreplacement). Whatistheprobabilitythat
   (i) thesumofthenumbersonthetwoballsselectediseven?
   (ii) Thenumberonthefirstballdrawnisevenifitisknowthatthesumofthe twonumbersiseven.
2. Inalargepopulation,itisobservedthat30percentofthepeoplethatareblack havecanerand25percentofthepeoplearenot blackhavecancer.Assumethat10 percentofthepopulationis black.Whatis theprobabilitythatapersonselectedat randomandfoundtohavecancerisnotblack.
3. A vaccineproducesimmunityagainstsmallpoxin 98percentof cases.Suppose that,inalargepopulation,20 percenthave beenvaccinated.Findthe probabilitythatapersonwhocontractssmallpoxhasbeenvaccinated,assuming thatavaccinatedpersonwithoutimmunity hasthesameprobabilityof contracting smallpoxasanunvaccinatedperson.
4. InafactorymachinesA,B,Cproducerespectively20,30and50percentofthe totalproduction.Oftheiroutput3,4,5percentrespectivelyaredefectiveitems. Anitemisdrawnatrandomfromthetotalproductionandisfounddefective. WhataretheprobabilitythatitwasproducedbymachineA,B,C?
5. Inafacultyofacertaincollege,60%ofthestudentsarefemale:20%ofthe femalesand50%ofthemalearestudyingmathematics.If astudentdatacardis selectedatrandomandthestudentisfoundtobestudyingmathematics,whatis theprobability thattheselectedstudentisamale?
6. InJAMBexaminationeachquestionhas5possibleanswers,exactlyoneofwhich iscorrect.Ifastudentknowstheanswerheselectsthecorrectanswer.Otherwise heselectsoneansweratrandomfrom the5possibleanswers.Supposethatthe studentknows theanswerto70%of thequestions.
   (i) Whatistheprobabilitythatonagivenquestionthestudentgetsthecorrect answer?
   (ii) Ifthestudentgetsthecorrectanswertoaquestion,whatistheprobability thatheknows theanswer?
   (iii) whatistheexpectedscoreof thestudentintheexamination?
7. Atelevisionsetretailerfindsoutthat80%ofthiscustomersbuycolouredT.V., andthat4outofevery20customerswhobuycolouredT.Vsetalsobuy antenna. Calculatetheprobabilitythat:
   (i) arandomlyselectedcustomerbuysantenna
   (ii) arandomlyselectedcustomers who buysantennahasboughta coloredT.V
   (iii) arandomlyselectedcustomerwhohasnotboughtanantennahasboughta

coloredT.Vset.

8. Inafactory,machinesA,BandCproduce20,30and40percentofthetotal output, respectively.Of theiroutput 5,4and3percentrespectivelyaredefective bolts. A bolt is chosen at random and found to be defective. What isthe probabilitythattheboltcamefrom machine(i)A (ii)B(iii)C?
9. Fourpercent ofan article manufactured byacompany are defective. All thearticlesproducedareregularlyinspectedandthosefounddefectivearerejected.It isfoundthatone outof everyeightof defectivearticleproducedismissedby the inspector,whileeverygood articlepassesinspection.Whatistheprobability thata customerbuysadefectivearticleproducedbythecompany?

3.2Independence

Thenotionofindependenceisabasictoolofprobabilitytheory.Considertossing a dietwice,andletA_1betheeventthat thefirsttoss gives an evennumber andA_2 theeventthatthesecondtossgivesanevennumber.Theevent$A_1 \cap A_2$ isthe eventthatbothtossesgiveevennumbers.TheoccurrenceofA_1 doesaffectthe probabilityofA_2occurring.

Therefore

$P(A_2|A_1)=P(A_2)$and$P(A_1\cap A_2)=P(A_1)P(A_2|A_1)=P(A_1)P(A_2)$. Hence,wesay thattwoeventsA_1andA_2areindependentif

$P(A_1\cap A_2)=P(A_1)P(A_2)$.

Example1.19

TossafairdietwiceandletAbetheevent“thefirsttossshows3”andBbethe eventthesum of thetwonumbersshowingis7.

$$P(A) = \frac{1}{6},$$

$$B \equiv \{(1,6), (6, 1), (2, 5), (3, 4), (4, 3), (5, 2)\}$$

$$P(B) = \frac{2}{36} = \frac{1}{6}$$

$$A \cap B \equiv (\textit{the first is 3 and sum is 7}) = \{(3,4)\}$$

Therefore,

$$P(A \cap B) = \frac{1}{36}$$

Thus $P(A\cap B)=P(A)P(B)$hence,A andBareindependentevents. Ingeneralcaseof n events,wehavethefollowing definition.

Definition2.6

TheeventsA$_1$,A$_2$,…A$_n$areindependentif

(i) $P(A_i \cap A_j) = P(A_i)P(A_j)$for allI≠j

(ii) $P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k)$ for all I, j, k such that $i \neq j \neq k$(2.7) (n-1)$P(A_1 \cap A_2 \cap A_3 \ldots A_n) = P(A_1 P(A_2)P(A_3)\ldots P(A_n)$

Thatis,theevents$A_1, A_2, \ldots A_n$aresaidtobemutuallyindependentif

$P(A_{n1} A_{n2} \ldots A_{nk}) = P(A_{n1})P(A_{n2})\ldots P(A_{nk})$ For everysubsequenceoftwoormoreevents.

$A_{n1}, A_{n2} \ldots A_{nk}$isasubsequenceof$A_1, A_2 \ldots A_n$ifthesubscriptintegerssatisfy

$1 \le n_1 < n_2 \ldots n_k \le n$.

If n=3,wehave,A_1, A_2, A_3areindependentif

(i) $P(A_1 \cap A_2) = P(A_1)P(A_2)$ $P(A_1 \cap A_3) = P(A_1)P(A_3)$ $P(A_2 \cap A_3) = P(A_2)P(A_3)$
(ii) $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

Condition(i) is calledpairwiseindependent.Wemightthinkthatpairwiseindependence always impliesindependence. But this is not necessarily so, as illustrated by the followingexample.

Example2.20

Letapairoffairdiceberolledonce.Consider theevents.A_1numberappearingonthe firstdieis even,A_2=thenumberappearingontheseconddieisodd={1,3, 5} andA_3= thedifferenceof thetwonumbersiseven

$A_1 \equiv$ {(2,1),(2,2),(2,3),(2,4), (2,5),(2,6),(4,1),(4,2),…,(6,1),(6,2),…} $A_2 \equiv$ {(1,1),(1,3),(1,5),(2,1),(2,3),(2,5),(3,1),…}

$A_3 \equiv$ {(1,1),(1,3),(1,5),(2,2), (2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4), (4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(5,6)}

$A_1 \cap A_2 \equiv$ {(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1), (6,3),(6,5)}

$A_2 \cap A_3 \equiv$ {(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1), (5,3),(5,5) $A_1 \cap A_2 \cap A_3 = \Phi$.

Thus

$P(A_1) = ½, P(A_2) = ½, (P(A_3) = ½$

$P(A_1 \cap A_2) =$ --,$P(A_1 \cap A_2) = \frac{9}{36}$,P≠$\frac{1}{4}$($A_2 \cap A_3$)= - $\frac{1}{4}$ $\frac{1}{4}$

$P(A_1 \cap A_2) = P(A_1)P(A_2; P(A_1 \cap A_3) = P(A_1)P(A_3); P(A_2 \cap A_3) = P(A_2)P(A_3)$ Thatis,A_1, A_2, A_3arepairwiseindependentbut

$$P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3) = -\frac{1}{8}$$

Sincecondition(ii)isnotsatisfied,weconcludethat$A_1, A_2$and A_3arenotindependent.

Definition2.7

If theevents$A_1, A_2, \ldots A_n$areindependentthen

$$P(A_1 \cap A_2 \ldots A_n) = P(A_1)P(A_2)\ldots P(A_n) \quad (2.8)$$

Example2.21

Amanfires10shotsindependentlyatatarget.What istheprobabilitythathehitsthe target(i) 10times;(ii)atleastonce
If hehas probability1/3of hittingthetargeton any givenshot.

(i) Let$A_i$bethe event “hehitsthetargetattheithshot”(I =1,2,3,…10)

$A_1 \cap A_2 \cap \ldots \cap A_{10}$istheeventhehitsthetarget10times$A_1, A_2, \ldots A_{10}$are independentevents.
Therefore,theprobabilityof hittingthetarget10timesis

$$P(A_1 \cap A_2 \cap \ldots, \cap A_{10}) = P(A_1)P(A_2)\ldots p(A_{10}) = \frac{1}{3}, \frac{1}{3} \ldots \frac{1}{3} = \left(\frac{1}{3}\right)^{10}$$

(ii) P(hittingthetargetatleastonce)= 1–P(not hittingthetargetatall).

$P(\text{not hittingthetargetatall}) = P(A_1 \cap A_2 \cap \ldots \cap A_{10})$

Where,$A_i \equiv$ nothittingthetargetattheithshot.$A_1, A_2, \ldots, A_{10}$areindependent eventsand $P(A_i) = 1 - \frac{1}{3} = \frac{2}{3}$.
Hence

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_{10}) = P(\bar{A}_1)P(\bar{A}_2)\ldots P(\bar{A}_{10}) = \frac{2}{3} \frac{2}{3} \ldots \frac{2}{3} = \left(\frac{2}{3}\right)^{10}$$

Examples

2.22 Supposeaboxcontains5redand3 blackballs.A ballischosenatrandomfrom theboxandthenasecond ball is drawnatrandomfromtheremainingballsinthe box.Findtheprobabilitythat

(i) bothballsareblack

(ii) bothballsarered

(iii) thefirst ball isblackandthesecondisred

(iv) thesecondisred

(v) thesecondis black.

Solution

LetA_1betheevent"thefirst ball isred
$\bar{A}_1$be the event "thefirst ballisblack
$\bar{A}_2$be theevent"thesecondballisred
$\bar{A}_2$be the event "thesecondballisblack

(i) P(bothballsareblack)=$P(\bar{A}_1 \cap \bar{A}_2)$

$$P(\bar{A}_1 \cap \bar{A}_2) = P(\bar{A}_1)P(\bar{A}_2 D \bar{A}_1)$$

$$P(\bar{A}_1) = \frac{3}{8} \quad P(\bar{A}_2 D \bar{A}_1) = \frac{2}{7}$$

Thus

$$P(\bar{A}_1 \cap \bar{A}_2) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

(ii) P(both balls are red) = $P(A_1 \cap A_2)$

$$P(A_1 \cap A_2) = P(A_1)P(A_2)DA_1)$$

$$P(A_1) = \frac{5}{8} \quad P(A_2 D A_1) = \frac{4}{7}$$

Thus

$$P(A_1 \cap A_2) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$$

(iii) $P(A_1 \cap A_2)$ = P(first ball is black and the second ball is red) = $P(\bar{A}_1)P(\bar{A}_2)\bar{A}_1)$

$$P(\bar{A}_1) = \frac{3}{8} \quad P(A_2 D \bar{A}_1) = \frac{5}{7}$$

Thus

$$P(\bar{A}_1 \cap A_2) = \frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$$

(iv) $P(A_2) = P(A_2 \cap A_1) + P(A_2 \cap \bar{A}_1)$

From(ii)and(iii)wehave $\frac{5}{14} + \frac{15}{56} = \frac{35}{56} = \frac{5}{0}$

(v) $P(\bar{A}_2)=P(A_1 \cap \bar{A}_2)+P(\bar{A}_1 \cap \bar{A}_2)=$53 + 3 - 3
0 7 26 0

2.23 Twofairdicearerolled.Giventhatthediceshow differentnumbers,what isthe probabilitythatatleastonedieshows a6?

Solution

LetAbetheevent:thediceshowdifferentnumbers.

A ≡{(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(1,5),(5,1), (1,6),(6,1), (2,3),…(5,6),(6,5)

B≡Atleastonedieshow a6.

≡ {(1,6), (6,1),(2,6),(6,2),(3,6),(6,3),(4,6),(6,4), (5,6),(6,5)}

A ∩ B= {(1,6), (6,1),(2,6),(6,2),(3,6),(6,3),(4,6),(6,4),(5,6),(6,5)}

P(BDA)= $\frac{10}{30} = \frac{1}{3}$

2.24 LetAandBbeanytwoeventsdefinedonthesamesamplespace.SupposeP(A)= 0.3andP(A B)=0.6.FindP(B)suchthat

(i) A and Bareindependent

(ii) A and Baremutuallyexclusive.

Solution

(i) If AandBareindependent,thenP(A ∩B)=P(A)P(B)

Thus

P(A ∪B)=P(A)+P(B)–P(A)P(B)

Wehave

0.6= 0.3+P(B)– 0.3P(B)= 0.3+0.7P(B)

P(B)= $\frac{0.3}{0.7}$ =0.43

(ii) If AandBaremutuallyexclusive,then

P(A∩ B)=0

Thus

P(A ∪ B)=P(A)+P(B)
0.6= 0.3+P(B)
P(B)=0.3

2.25 TwowomenA,andBshareanofficewithasingletelephone.Theprobabilitythat anycallwillbeforAis $\frac{2}{3}$.SupposethatAisoutofherofficeduringtheofficehourshalfofthetimeandBonethird. Findtheprobabilitythatforanycallduring theworking hours

(i) nooneisintoanswer thecall

(ii) A callcanbeansweredbythepersonbeingcalled

(iii) Twosuccessivecallsareforthesamewoman

(iv) AcallerwhowantsAhastotrymorethantwotimestogether.

Solution

(i) P(AandBarenotin theoffice)= $\frac{1}{2}\frac{1}{3} - \frac{1}{6}$

(ii) P(Acallcanbeansweredbythepersonbeingcalled)

=P(callforA and Aisintheoffice)+P(callfor BandBisin theoffice)

= $\frac{2}{3}\frac{1}{2} + \frac{1}{3}\frac{2}{3} - \frac{5}{9}$

(iii) P(for AA)+P(forBB)= $\frac{2}{3}.\frac{2}{3} + \frac{1}{3}\frac{1}{3} = \frac{5}{9}$

(iv) P(X>2) =1–P(X=1)–P(X =2).

WhereX number oftimefor acallerwhowantsA hastotry together. Thus, P(X>2) =1-

$$\left(\frac{2}{3}\frac{1}{2}\right) - \left(1 - \frac{2}{3}\frac{1}{2}\right)\left(\frac{2}{3}\frac{1}{2}\right) = 1 - \frac{1}{3} - \frac{2}{9} = \frac{4}{9}$$

SELF ASSESSMENT EXERCISE3

1. (a) Show thatifA andBareindependentevents,then

 (i) A and B^c, (ii)A^candB^carealsoindependent.

2. LetAandBdenotetwoindependentevents suchthatA isasubset of B.provethat eitherP(A)=0orP(B)= 1.

3. Amanfires10shotsindependentlyatatarget.Theprobabilityofhittingthetarget atany shotis 1/3.Calculatetheprobabilitythat
 (i) noneoftheshotshitsthetarget
 (ii) Atleastoneshothitsthetarget
 (iii) Thetargetishit atleasttwiceifitis knowthatitishitatleastonce.

4. Aboxcontains6redballsand4whiteballs.Threeballsaredrawnfromthebox oneaftertheotherwithoutreplacement.Find theprobability that
 (i) thefirsttwo arewhiteandthethirdred
 (ii) thefirsttwo arewhiteandthethirdwhite.
 (iii) twoareredand one iswhite
 (iv) thesecondballdrawn isred
 (v) thethirdballdrawniswhite

5. A dieisrolled8times.Whatistheprobabilitythat
 (i) exactly2 sixesappear.

(ii) atleast2 sixesappear.
(iii) atmost2sixesappear.

6. ProvethatifA$_1$,…,A$_n$areindependenteventsthen
   (i) $P(A_1 \cup A_2 \cup \cdots \cup A_n)=1-[1-P(A_1)\}\{1-P(A_2)]\ldots[1-P(A_n)]$
   (ii) $P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq 1- e^{-[P(A_1) + P(A_2)+ \ldots + P(A_n)]}$

   Hint:$(1-x_1)(1-x_2)\ldots(1-x_n) \geq e^{-(x_1 +x_2+\ldots+x_n)}$ $x_i \leq 1$
7. Show that$P(A\cap B\cap C)=P(ADB\cap C)P(BDC)P(C)$.
8. Adieistossedntimes.Whatistheprobabilitythata6appearsatleasttwotimesinthentosses.
9. Suppose thatAorBoccursis 0.7whileP(A)=0.2,findP(B).
10. Aboydecidestocontinuetossingafaircoinuntilhehasthrowntotalofthree heads.FindP$_n$,theprobability thatexactlyntosseswillneeded.
11. Six blood samples are selected from 40 blood samples, of which four are cancerous.Whatistheprobabilitythatexactlytwooftthebloodsamplesselected arecancerous?
12. ProvethatifA$_1$,A$_2$,…A$_n$areany n events,then

    $P(A_1\cap A_2\ldots\cap A_n)>1-\{P(A_2^c)+ P(A_2^c)+\ldots+P(A_n^c)\}$.

13. Provethat
    (i) $P(A\cap B^c)=P(A)-P(A \cap B)$
    (ii) $P(A\cap B^c)=1-P(A) -P(B)\ \ P(A \cap B)$ (iii) $P(A)= P(A\cap B)+P(A \cap B^c)$
    (iv) $P(A\cap B)\geq P(A)+ P(B)-1$.
14. three coinshaveprobability0.5,0.6and0.8forheadsrespectively.One of thenis selectedatrandom,thatis,withequalchanceforeach,andtossed.Iftheoutcome is head,whatisthe probabilitythatthecoinwithprobability 0.8for headswas selected?
15. On acertainweekendthereare4movies.Calculatetheprobabilitythat atleastone of A andBwillbeselectedbyoneormoreof the3students.
16. A box containsnwhiteballsnumbered1ton,nblackballsnumbered1ton,and n red ballsnumbered1 to n.iftwo ballsaredrawn atrandomwithoutreplacement, whatisthe probabilitythatbothballs willbeof thesamecolouror bear thesame numbers.
17. Onthefirstround,threefaircoinsareflippedatrandom.Thecoinsresultingin heads areflippedatrandom onthesecondround. Ifthesecondroundresultsin exactlyonehead,whatistheconditionalprobabilitythatthefirstround ended inexactlytwohead?

4.0 CONCLUSION

By now the idea of using Bayes' Theorem for calculating probability of mutually exclusive events has been dealt with together with calculation of independent events.

5.0 SUMMARY

Recall that in this unit we have studied the calculation of probability of mutually exclusive events using Bayes' theorem. We also established the calculation of the probability of independent events.

6.0 TUTOR-MARKED ASSIGNMENT

1. State the number of different arrangements or permutations consisting of 3 letters each which can be formed from the 7 letters A,B,C,D,E,F,G.

2. A student tossed a fair coin until he has thrown total of four heads. Find P_n, the probability that exactly n tosses will be needed.

3. A basket contains 4 red balls and 6 white balls. Three balls are drawn from the box one after the other without replacement. Find the probability that
   (i) Two are red and one is white
   (ii) The third ball drawn is white
   (iii) The first two are white and the third red

7.0 REFERENCE/FURTHER READING

Harry Frank & Steven C. Althoen (1995). Statistics: Concepts and Applications. Cambridge University Press.

UNIT3 DISCRETE RANDOM VARIABLES

CONTENT

1.0 Introduction
2.0 Objectives
3.0 Main Content
3.1 Discrete Random varianles
3.2 Properties of the probability distribution function
3.3 Special discrete random variables
3.4 Bernoulli
3.5 Trail The Binomial
3.6 The Binomial Random variable
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Reading

1.0 INTRODUCTION

A random variable is a variable whose actual numerical value is determined by chance.There aretwoeasilyidentifiabletypesofrandomvariables,discreteand continuous.A discretevariableisone thattakesonlya limitednumberofpossible values,otherwisethe variableiscalledcontinuous. This unitisdevotedtodiscrete randomvariables.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the probability density function of a sample space
- state the properties of the district random variables
- solve the probability density function with replacement and without replacement
- statethe Bernoulli Trail, Bernoulli random variable.

3.0 MAIN CONTENT

3.1 DiscreteRandomVariable

Random variable which takes on a finite or countable infinite number of values is called a discrete random variable which one which takes on a noncountably infinite number of values is called a nondiscrete or continuous random variable.

Example3.1

ConsidervariableX, the numberof heads inthreetossesof a coin. Therearefour possiblevalues(0,1,2,3,) of X.

TheactualvalueassumedisduetochancethereforeXisarandomvariable.The samplespacefor thisexperimentis

Ω= {HHH, HHT,HTH,THH,HTT,THT,TTH,TTT}
X =0iftheoutcomeisTTT
X =1iftheoutcomesisHTT,or THT,orTTH
X =2iftheoutcomeisHHTor HTHor THH
X=3iftheoutcomeisHHH

Letpbetheprobabilityofthecoinlandingtail.Sincelandingtailandlandingheadare exhaustiveeventstheprobabilityof thecoinlandingheadis1-p.

P(T∩ T∩T)=P(T)P(T)P(T)

Sincetheoutcomeateachtrialsareindependent.

P(TTT)= p.p.p=P^3

P(HTH)=P(T)P(H)P(H)=p(1-P) (1-P) =$(1-P)^2$

P(HTH)=P(H)P(H)P(H)=(1-P) (1-P)= $P(1-P)^2$

P(HHT)=P(H)P(H)P(T)=(1-P)(1-P)P==$(1-P)^2$

Theprobability ofgettingtwoheads=P(THH)+P(HTH)+P(HHT)= $3P(1-p)^2$

Similarlywehave

P(ohead)=P(TTT)=p^3

P(1head)=P(HTT)+P(TTH)+P(THT)=$3p^2(1-p)$
P(3heads)=P(HHH)=$(1-p)^3$

Thuswehavethefollowingtable

Heads	0	1	2	3
Probability	P^3	$3p^2(1-p)$	$3p(1-p)^2$	$(1-p)^3$

P(0head)=P(X is 0)=p^3,p(X=1)$3p^2(1-p)$andsoon

Therandom variableXdefinedaboveisanexampleof whatiscalledadiscreterandom variable.ArandomvariableisdenotedbyacapitallettersuchasX.Y,Z…andthe valuesthattherandom variabletakesonisdenoted byalowercaseletterx,y,z… Thenotation(P(X=x)meanstheprobabilitythat therandom variableX takesonthevalue x.

Definition3.1

ArandomvariableXonasamplespaceΩisafunctionassignedtoeachelementΩone andonly onerealnumberX (Ω) =x,thespaceof X isthesetof realnumber
Φ ={x:x=x(Ω),Real (Ω)}.

Definition3.2

Arandomvariablexisdiscreteif itcanassumeatmostafiniteoracountableinfinite numberofpossiblevalues.
Intheaboveexample,

Ω= $\{w_1, w_2. W_3, \ldots\ldots w_n.\}$

Wherew_1= HHH,w_2=HHT…,w_8= TTT

={0,1,2,3}andX (w_1)= 3,X(w_2)=2.X(w_3)= 2,X(w_4)=2

X (w_5= 1,X(w_6)=1,X (w_7=1,X (w_8)=0

Thatis{w:X(w)=x_1}id anevent

Definition3.3

The real valued function f defined on R by f(x) = p(X = x) is called the discrete probabilitydensityfunctionof X.

LetXbeadiscreterandomvariableandsupposethatthevaluesitcanassumearex_1, $x_2 \ldots, x_n$

Theprobability canbewrittenas

$P(X= x_1)=f(x_1),=P(X=x_2)= f(x_2)\ldots,P(X =x_n)=f(x_n)$ Suchthat
F(x) iscalled probabilitydensityfunctionof X

For anillustration,letus considerthefollowingexamples.

Example3.2

Supposeapairoffairdiceistossedonce,LetX,Y,Zrepresentthesum,maximum and minimumrespectivelyofthetwonumbersappearingfind theprobabilitydensityfunction of

(i)X,(ii)Y, (iii)Z.

Solution

ThesamplespaceΩ={(1,1),(1,2),…, (6,6)} consistof36elements

(i) $P(X =2)=P\{(1,1)\}= \frac{1}{36}$

$P(X =3)=P\{(1,2), (2,1) = \frac{2}{36}$

$P(X =4) =P\{(1,3),(3,1) ,(2,2)\} = \frac{3}{36}$

$P(X =12)=P\{(6,6)\}= \frac{1}{36}$

Thus, $F(2)= \frac{1}{36}$, $F(3) = \frac{2}{36}$, $F(4)= \frac{3}{36}$

Intabularformwehave

X	2	3	4	5	6	7	8	9	10	11	12
F(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

infunctionform

(ii) Y =Maximumof thetwo number
PossiblevaluesofYare1,2,3,4,5,6

g(1)= P(Y=1)=P{(1,1)}=1/36

g(2)= P(Y= 2) =P{(1,2),(2,1) (2,2)= 3/36

g(3)= P(Y=3)=P(1,3),(3,1),(2,3) (3,2) (3,3)} = 5/36 g(4)= P(Y=$)=P{(1,4)(4,1) 2,4) (4,2) (3,4) (4,3) (4,4) = 7/36

g(5)= P(Y=5) =P{(1,5), (5,1),(2,5) (5,2) (3,5) (5,3)(4,5)(5,4)(5,5)= 9/36

g(6)= P(Y=6) =P{(1,6),(6,1) (2,6) (6,2)(3,6)(6,3)(4,6) (6,4) (5,6) (6,5)(6,6) = 11/36

Puttinginforma tablewehave

Y	1	2	3	4	5	6
g(y)	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Thiscan bewritteninafunctionalformas

(iii) Z= Minimum of thetwo numbers

ThepossiblevaluesofZare1,2,3,4,5,6

P(Z-1)= P{(1,6), (6,1)(1,5)(5,1)(1,4)(4,1) (1,3)(3,1)(1,2)(2,1)(1,1)= $\frac{11}{36}$

P(Z=2)=P(2,6)(6,2)(2,5)(5,2)(2,4(4,2) (2,3)(3,2) (2,2) = 9/36
P(Z=4)P{(4,6)(6,4)(4,5) (5,4) (4,4) = 5/36

P(Z=5)=P(5,6) (6,4) (5,5) = 3/36

P(Z=6) =P(6,6) = 1/36

Puttinginformofatablewehave

Z	1	2	3	4	5	6
h(z)	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

H(Z)= $\frac{13 - 2z}{36}$ z=1,2,3,4,5,6

=0for othervaluesofx.

The probability density function of a discrete random variable X has the following properties

(i) O $\leq$f(x) $\leq$1,xER
(ii) {X:f(x) =0} isafiniteor countableinfinitesubsetof R
(iii) $\sum f(x_i)=1$

Example3.3

LetX,Y,Zbetherandom variableintroducedinexample3.1above

The above three properties are satisfied, properties (i) &(ii) are immediate from definitionof probabilities.Tocheck(iii)wehave

$\sum f(x_1)$=1/36+ 2/36+3/36+ 4/36+ 5/36+6/36+5/36+4/36+3/36+2/36+1/36=1

$\sum G(y_i)$= 1/36+3/36+5/36+7/36+ 9/36 +11/36=1

$\sum h(z_i)$=11/36+9/36+7/36+5/36+ 5/36+3/36 +1/36=1

Example3.4

Supposeaboxcontainsballsofwhich4areredand6areblack.Arandomsampleof size3isselected.LetX denotethenumberofredballsselected.Find theprobability densityfunctionofxif

(i)Samplingiswithoutreplacement

(ii)Samplingiswithreplacement

(iii) ThepossiblevaluesofXare0,1,2,3.

P(X= 0) =(Pnoredball in thethreeballsselected)$=\frac{{}^{6}C_{3}}{{}^{10}c_{5}}=\frac{6}{9}\frac{5}{8}\frac{4}{6}=\frac{1}{6}$

P(X= 1) =P(1redand2blackballs)$=\frac{{}^{4}C_{1}\times{}^{6}C_{2}}{{}^{10}C_{3}}=\frac{1}{2}$

$P(X=2)=\frac{{}^{4}C_{2}\times{}^{6}C_{1}}{{}^{10}C_{3}}=\frac{3}{10}$

$P(X=3)={}^{4}C_{3}/{}^{10}C_{3}=\frac{1}{30}$

Thusthep.d.fis

X	0	1	2	3
F(x)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

As acheck,weaddupalltheprobabilities

1/6+½ +3/10+1/30=1

(ii) Samplingwith replacement

P(X= 0) =P (firstballis black,secondblackandthethirdblack)=P(bbb). Theprobability ofblackatany drawingis
6/10=3/5

Thisvalueisconstantsincedrawingiswithreplacement.Therefore

P(bbb) = $\left(\frac{3}{5}\right)^3$ $\frac{3}{5}\frac{3}{5}\frac{3}{5}=$

P(X= 1) =P(Rbb)+ P(bRb)+P(bbR)= $\frac{2}{5}\frac{3}{5}\frac{3}{5}+\frac{3}{5}\frac{2}{5}\frac{3}{5}+\frac{3}{5}\frac{3}{5}\frac{2}{5}$

$=\left(\frac{2}{5}\right)$ $3.\frac{2\times 3\times 3}{5\times 5\times 5}=3$

$\left(\frac{3}{5}\right)^2$ Similarly,

P(X= 2) =$\left(\frac{3}{5}\right)$ $\frac{2}{5}\frac{2}{5}\frac{3}{5}+\frac{2}{5}\frac{3}{5}\frac{2}{5}+\frac{3}{5}\frac{2}{5}\frac{2}{5}=3\times\left(\frac{2}{5}\right)^2$

P(X=3) =P(RRR)=$\left(\frac{2}{5}\right)^3$ $\frac{2}{5}\frac{2}{5}\frac{2}{5}=$

Thus,thep.d.fofXis

X	0	1	2	3
f(x)		$\left(\frac{3}{5}\right)^3$	$\left(\frac{3}{5}\right)$	

Thiscan bewrittenas

$$f(x)=\left(\frac{2}{5}\right)^{3-x} \quad x=0,1,2,3 \begin{cases}\left(\frac{2}{5}\right)^x & \\ 0 & \text{elsewhere}\end{cases}$$

Example3.5

A box contains6 ballslabeled1,2,3,4,5,6twoballsaredrawn atrandomoneafter theother.LetX denotethelargerofthetwo numbersontheballsselected,obtainthe probabilitydensityfunctionof X if
(i)Samplingiswithoutreplacement (ii)Samplingiswith replacement

Solution

(i) ThepossiblevaluesofXare2,3,4,5,6.Thelargerofthetwonumberscannotbe 1sincesamplingiswithoutreplacementandwecannotgetsamenumbertwice.

P(X= 2) =P{(1,2),(2,1)}= $\frac{4}{6.5}$

P(X= 3) =P{(1,3),(3,1),(2,3), (3,2)} = $\frac{4}{6.5}$

P(X=5)=P{(1,5), (5,1),(2,5),(5,2),(3,5),(5,3),(5,4),(4,5)}= $\frac{8}{6.5}$

Similarly,
P(X= 6) = $\frac{10}{6.5}$

Thustheprobabilitydensityfunction ofXis

X	2	3	4	5	6
f(x)	1/15	2/15	1/5	4/15	1/3

Infunctional,form,this canbewrittenas

$$f(x) = \begin{cases} \frac{2(x-1)}{6.5} = \frac{x-1}{15} & x = 2,3,4,5 \\ 0 & \text{elsewhere} \end{cases}$$

(ii) ThepossiblevaluesofX are1,2, 3,4, 5, 6.Thelargercanbe1inthiscase since(1,1) isapossibleoutcome. Thesamplespaceconsists36possible outcomes

P(X= 1) =P(1,1) = $\frac{36}{}$

P(X= 6) =P{(1,6),(6,1),(2,6), (6,2).,,,(6,6)} = $\frac{11}{36}$
HencetheprobabilitydensityfunctionX is

X	1	2	3	4	5	6
f(x)	1/36	3/36	5/36	7/36	9/36	11/36

Infunctionalform wehave

f(x) = $\frac{112x-1}{36}$ x=1,2,3,4,5,6

3.6 Acoupledecidesthattheywillcontinuetohavechildrenuntileitherthey haveaboy andagirlinthefamilyortheyhavefour children.Assumingthatboys andgirlsareequally likelytobeborn.LetXdenotethenumberofchildreninthe family.Findtheprobability densityfunctionofX.

Thenumberof childreninthefamilycaneitherbe2,3,or 4.

P(X= 2) =P{(B,G), (G,B)}

Thatis,thecouplewillstophavingmorechildrenifthefirstchildisaboyandthesecondof girl or thefirstisagirland thesecondaboy.

Thus,

P(X= 2) =P(B,G)+P(G,B)=P(B)P(G)+P(G)P(B)

$$\left(\frac{1}{2}\right. \quad \left(\frac{1}{2} \times \frac{1}{2}\right) + \quad \times \frac{1}{2}\Big) = \frac{1}{2}$$

Similarly,

P(X= 3) =P{(BBG),(GGB)}=P(BBG)+P(GGB)= $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

Hence,theprobabilitydensityfunctionofX is

X	2	3	4
f(x)	½	¼	¼

SELF-ASSESSMENT EXERCISE1

1. Afaircoinistosseduntilaheadorfivetailsoccur.LetXdenotethenumberof tossesof thecoin.ComputetheprobabilitydensityfunctionofX.

2. Aboxcontains2redballsand3blueballs.Ballsaresuccessivelydrawnwithout replacementuntilablueballisdrawn.LetXdenotethenumberof drawsrequired. Computethep.d.fof X

3. Thepdfof arandomvariableX isgivenby

X	1	3	4	6	8
f(x)	K	k	k	k	k

Find(i)K(ii)P(X≥3).

4. Tossafaircointwotimes.LetXbethenumberofheadsobtained.FindthepdfofX.
5. Acoinwithprobabilitypofaheadistosseduntilaheadappears.LetXdenotethe numberof times thecoinistossed. Findthepdfof X.
6. Afairdieistossedtwice.LetXdenotetheproductofthetwonumbersappearing. Findthepdfof X
7. Afaircoinistossed3times.LetXrepresentthedifferencebetweenthenumberof headsand thenumberof tailsobtained.Find thepdfof X.
8. Thepdfof arandomvariableX isgivenbyf(x) ={k2^X,x=1,2,3,…N,zero elsewhere}.Findthevalueof K.

Definition3.3

ProbabilityDistributionfunction.LetXbearandomvariablewithprobabilitydensity functionf(x).TheprobabilitydistributionfunctionofX denotedf(x) is definedby F(x) =p(X ≤x) for xreal.

$=\sum_{y\le x} f(y)$

PropertiesoftheProbabilitydistributionfunction

1. Fisanondecreasingfunction,thatisifa<b,then F(a)<F(b).
2. lim F(b)=1. b-∞
   lim F(b)=0 b-∞
3. Fisright continuous.ThatisF(b+) =F(b).

LetAbeanysubsetofRandletf(x)betheprobabilitydensityfunctionofX.ewcan computeP(X εA) bynotingthat{w:X (w) εA} isaneventandthat {w:X(w)εA} =U{w :X(w) =x_i}.

X_iεA
Thus
P(XεA) =$\sum_{X_i\varepsilon A} f(X_i)$.

If Ais an intervalwithendpointsaandb,sayA =[a,b].

Then

{(Xε A) =P(a≤X ≤b)=$\sum_a^b f(x_i)$.

Example3.7

ConsidertherandomvariableXthesumofthetwonumbersappearingwhenafairdieis tossedtwiceof Example3.2.TheprobabilitydistributionfunctionforX is givenby

X	2	3	4	5	6	7	8	9	10	11	12
F(x)	1	3	6	10	15	21	26	30	33	35	36

SupposewewishtofindtheprobabilitythatXisbetween4and9(4and9inclusive)we writeitas

P(4≤X ≤9)= P(X =4) +P(X = 5)+P(X= 6) +P(X= 7)+P(X =8) +P(X =9)

$= \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + 36$

Thiscan alsobewrittenasP(X≤9)– P(X ≤3)= = F(9) – F(3) $= \frac{30}{36} - \frac{3}{36} = \frac{27}{36}$ $\frac{27}{36} = \frac{3}{4}$

3.2 SpecialDiscreteRandomVariables

3.2.1 InSection3.1weintroduceddiscreterandomvariablesand theprobability density functionofsome random variablesweredetermined.Somevariablesaresocommonand importantthatnamesaregiventothem. Inthissectionweshallconsiderinconsiderabledetailanumberofimportantdiscrete randomvariables.

BernoulliRandomvariables

Definition3.4

BernoulliTrail:

Arandomtrailor experimentiswhichtheoutcomecanbeclassifiedintooneoftwo mutuallyexclusiveandexhaustivewaysusuallycalled successorfailureiscalled a Bernoullitrail.Therandom variableassociatedwith BernoullitrailiscalledaBernoulli random variable(X).letX=0ifoutcomeisafailureandX =1iftheoutcomeisasuccess. Thatis any variableassumingonly twovaluesis calledaBernoullirandomvariable.

Suppose thatwetossacoinonce.Lettheprobabilityofitlandingheadbep.p= $\frac{1}{2}$, ifthe coinisfairandletdenotetheoutcomeofthetoss.ThentherearetwopossiblevaluesforX,Heads ortails.Thesetwovaluesaremutually,exclusiveandexhaustiveandwemayassociatethetwo possibleoutcomesofthetosswithvalues1,0oftherandomvariables

X.ThatX =1whenaheadappearsandX =0whenatailappears.

$P(X= 1) = p, P(X=0) = 1-p$.

Thep.d.f.of X is

X	0	1
f(x)	1–p	P

Or infunctionalform

$$F(x) = \begin{cases} p^X(1-p)^{1-x}, & x=0,1 \\ 0 & \text{elsewhere} \end{cases} \qquad 3.1$$

f(x)asdefinedaboveiscalledtheBernoulliprobabilitydensityfunctionandany variable X having(3.1)hasitsprobabilitydensityfunctioniscalledaBernoullirandom variable andissaidtohavetheBernoullidistribution.

3.2.2 TheBinomialRandomVariable

Thisisoneof themost importantrandomvariablesinstatisticsand themostimportant discreterandomvariable.ConsidernindependentrepetitionsofBernoullitrails.LetX_i,I= 1, 2,…, n be Bernoulli random variables associated with the trails. The randomvariables$X_1, X_2, \ldots X_n$ areindependentBernoullirandomvariables.Letusassumethe probabilityofsuccessisp and failure1-pand

$P(x_1=1) = p$

Then,sum$S_n = +x_2 + \text{---} + X_n$isthenumberofsuccessesinnBemoullitrials.Thatis,S_n is a counting-variablecounting the numberofsuccessesin nrepeatedtrials.Thisrandom variable$S_n$iscalledtheBinomialrandomvariable.ThepossiblevaluesofS_n are0,1,2,3,4,----,n.

$P(S_n=0) = p(\text{nosuccess})$

=p(1sttrailisafailure)P(2^{nd}trailisafailure)…P(nthtrailisafailure).

$= P(X_1=0)\, P(X_2= 0) P(X_3= 0) \ldots P(X_n= 0)$

$= (! -p)(1-p)(1-p)\ldots(1-p) = (1-p)^n$.

$S_n$is1ifthesequenceofoutcomeis

1000000…0or010…0 or 0.010…0,…0000001

$P(S_n=1)= P(1,0,0,0,\ldots0) +P(),1,0,0,\ldots,0) +\ldots+P(0,0,\ldots1)$

$=P(X_1=1)P(X_2=0)\ldots P(X_n=0) +P(X_1=0)P(X_2=1)\ldots P(X_n=0)+\ldots+P(X_1=0)\ldots P(X_n=1)$
$=P(1–p)\ldots(1– p)+(1-1)P(1–p)\ldots(1– p)\ldots(1–p)+(1–p)(1–p)$

$=P(1-p)^{n-1}+P(1-p)^{n-1}+\ldots+P(1-p)^{n-1}= np(1-p)^{n-1}$

Similarly,
$P(S_n=2)= {}^nC_2P^2(1-p)^{n-2}$

Where${}^nC_2$ isthenumberofsequencesinwhichexactly2havevalue1andtheothers0 e.g. (1,1,0,0,0,0,0,…,0),(1,0,1,0,…,0)…
Ingeneral,itcaneasilybeseenthat

$P(S_n=k)= {}^nC_2p^k(1-p)^{n-k},k= 0,1,\ldots,n$

Where${}^nC_2$isthenumberof sequencesinwhichexactlykhavevalue1andothers0. For examplewhenn=4,possiblesequencesofoutcomesaregivenbelow.

Sequence	S_n	$P(S_n)$
(0,0,0,0)	0	$(1–p)^4$
(1,0,0,0)	1	$P(1–p)^3$
(0,1,0,0)	1	$P(1–p)^3$
(0,0,1,0)	1	$P(1–p)^3$
(0,0,0,1)	1	$P(1–p)^3$
(1,1,0,0)	2	$P^2(1– p)^2$
(1,0,1,0)	2	$P^2(1– p)^2$
(1,0,0,1)	2	$P^2(1– p)^2$
(0,1,1,0)	2	$P^2(1– p)^2$
(0,0,1,1)	2	$P^2(1– p)^2$
(0,1,0,1)	2	$P^2(1– p)^2$

(1,1,1,0)	3	$P^3(1–p)$
(1,1,0,1)	3	$P^3(1–p)$
(1,0,1,1)	3	$P^3(1–p)$
(0,1,1,1)	3	$P^3(1–p)$
(1,1,1,1)	4	P^4

Thus, $P(S_n=0)= (1–p)^4$

$P(S_n=1)= 4p(1– p)^3$

$P(S_n=2)= 6p^2(1– p)^2$

$P(S_n=3)= 4p^3(1-p)$ $P(S_n=4)= p^4$

Theorem3.1

LetS_n denotenumberofsuccessesinnrepeatedBernoullitrails,withprobabilityof successp. theprobabilitydensityfunctionof$S_n$isgivenby

$f(x) =P(S_n= x) ={}^nC_x p^x(1– p)^{n-x}$ x=0,1,…,n (3.2)

0

Definition3.5

AdiscreterandomvariableXdenotingtotalnumberofsuccessesinntrailsissaidto havethebinomialdistributionif

$$P(X= x) = {}^{n}C_{x}p^{x}(1-p)^{n-x};x=0,1,2,\ldots,n$$
$$0\leq p\leq 1$$

Theconditionsunderwhichbinomialdistributionwillariseare

(i) Thenumberof trailsisfixed
(ii) Thereareonlytwo possibleoutcome‘success’or ‘failure’ateachtrial.
(iii) Thetrailsareindependent
(iv) Theprobability pof successateachtrailisconstant
(v) Thevariableisthetotalnumberofsuccessesinntrails.

Example3.8

A soldierfires10independentlyatatarget.Findtheprobabilitythathehitsthetarget.
(i) once
(ii) atleast9times
(iii) atmosttwotimes.

Ifhehasprobability0.8ofhittingthetargetatanygiventime?LetXdenotethenumber of timeshehitsthetarget.ThenX isabinomialvariablewithn=10andp= 0.8
Fromequation(3.2),wehave

$$P(X= x) = {}^{10}C_{x}(0.8)^{x}(0.2)^{10-x}$$

(i) $P(X= 1) = {}^{10}C_{x}0.8(0.2)^{9}= 8(0.2)^{9}$
(ii) $P(\text{Hehitsthetargetat least9times})=P(X\geq 9)$
$=P(X =9) +P(X =10)$

$P(X= 9) = {}^{10}C_{9}(0.8)^{9}(0.2) =10\times(0.8)^{9}\times 0.2=2(0.8)^{9}$

$P(X= 10)= {}^{10}C_{10}(0.8)^{10} = (0.8)^{10}$

Hence,

$P(X\geq 9) =2(0.8)^{9}+(0.8)^{10}= (0.8)^{9}(2+0.8)=(0.8)^{9}(2.8)= 0.3758$

(iii) $P(\text{atmosttwice})=P(X\leq 2) =P(X= 1) +P(X=2)$

$P(X= 0)=(0.2)^{10};$

$P(X= 1) =8(0.2)^{9};$

$P(X= 2) =45(0.8)^{2}(0.2)^{8}$

Thus,

$P(X\leq 2) =(0.2)^{10}+ 8(0.2)^{9}+ 45(0.8)^{2}(0.2)^{8}=0.00008.$

Example3.9

Afairdieisrolledfourtimes.Findtheprobabilityofgetting2sixes.Letuscallasixasuccessonat ossofadieandletXbenumberofsixes(successes)in4trails.Xis binomialrandom variablewithn=4 andp=1/6. Thus,

$P(X=2) = {}^4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = \frac{25}{216}$.

Example3.10

Supposethatacertaintypeofelectricbulbhasaprobabilityof0.3offunctioningmore than800hours.Outof50bulbs,whatisprobabilitythatlessthan3willfunctionmore than 800hours. Let X be the number of bulbs functioning more than 800 hours. AssumingthatXhasaBinomialdistribution,

$P(X=x) = 50C_x(0.3)^x(0.7)^{50-x}$

$P(X<3) = P(X=0)+P(X=1)+P(X=2)$

$P(X=0) = (0.7)^{50}; P(X=1) = 50.(0.3)(0.7)^{49}$

$P(X=2) = {}^{50}C_2(0.3)^2(0.7)^{48}$.

Thus,

$P(X<3) = (0.7)^{50} + 15(0.7)^{49} + 110.25(0.7)^{48} = 0.0000046$

SELF ASSESSMENT EXERCISE2

1. Afaircoinistossed4times.Computetheprobabilitythat(i)exactlytwoheads occur(ii)atleast3 headsoccur.
2. Aninvestigationrevealsthatfouroutofeveryfivepatientsarecuredofmalaria whentreatedwithanewdrug.Ifasampleoftenpatientsistreatedbythenew drug,computetheprobabilitythat(i)exactlysixpatientsarecured,(ii)atmost fourpatientsarecured
3. Amanfires12shotsindependentlyatatarget.Theprobabilityofhittinghistarget is¼.
   (i) Whatistheprobabilityofhishittingthetargetatleasttwotimes?
   (ii) Howmanytimesmusthefiresothattheprobabilityof hishittingthetarget atleastonceis greaterthan7/9?
4. Six childrenareborninahospitalinagivenday.Calculate
   (i) theprobability thatthenumberofboys isthe sameasthenumberofgirls.
   (ii) theprobability thattherearemoregirlsthanboys
   (iii) themostlikelynumberof boys.

4.0 CONCLUSION

Unitintroducesideaof arandomvariableand itsprobabilitydensity function.

Section3.2-3.4aredevotedtosomespecial discreterandomvariables.Bernoulli, BinomialPoisson, uniform,andnegativebinomial.

5.0 SUMMARY

The unit had treated the following:

- Discrete Random variables
- Properties of the probability distribution function
- Special discrete random variables
- Bernoulli Trail
- The Bernoulli Random variable

6.0 TUTOR-MARKED ASSIGNMENT

1. Anexperimenthas90percentprobabilityofsuccessand10percentprobabilityof failure.Theexperimentisrepeatedfour times.Findtheprobabilityofobtaining
   (i) Nosuccess
   (ii) Nofailure
   (iii) Twosuccessesandtwofailures.

2.LetXbeaBinomialrandomvariablewithparaments(n.p).show that
   (i) P(X= k) firstincreasesmonotonicallyandthen decreases monotonically.
   (ii) ThevalueofkthatmaximizesP(X=k)isthelargestintegerlessthanor equalto(n+1)P
   (iii) P(X=k) = $P(X-1)\cdot\frac{n-k-1}{k}\cdot\frac{p}{1-p}$ $= k -$

3. Afaircoinistossedrepeatedlyuntilthreeareobtained.FindP_n,theprobability thatexactlyntossesareneeded.

4. Show that(i) $\sum_{x=0}^{n} {}^{n}C_{x}p^{x}(1-p)^{n-x}=1$

   (ii) $\sum_{x=0}^{n} x\,{}^{n}C_{x}p^{x}(1-p)^{n-x}=np$
   $0<p<1$.

7.0 REFERENCE/FURTHER READING

Harry Frank & Steven C. Althoen (1995). Statistics: Concepts and Applications. Cambridge University Press.

UNIT 4 GEOMETRIC RANDOM VARIABLE

CONTENT

1.0 Introduction

2.0 Objective
3.0 Main Content
    3.1 Geometric Random Variable
    3.2 Pascal or Negative Binomial Random Variable
    3.3 The Hyper geometric Random Variable
    3.4 The Poisson Random Variable
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 Reference/Further Reading

1.0 INTRODUCTION

AdiscreterandomvariableXiscalledageometricrandom variableif itsprobability densityfunctionisgivenby

$(1-p)^{x-1}p$ x=,1,2,3…

F(x) =p(X=x)= 0 otherwise $0<p<1$ (3.3)

WhereXisthenumberofindependentBernouillitrialstakenforthefirstsuccessto occur.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- calculate the success or failure using binomial expansion
- describe the use of geometric random variable for probability problems
- use binomial distribution for calculation
- distinguishbetween Pascal distribution and geometric distribution.

3.0 MAIN CONTENT

3.1 GeometricRandomVariable

Consider a Bernoulli trail with probability p of a success on one trail. The trailis continueduntilasuccessoccurs.LetXdenotenumberoftrailsbeforethefirstsuccess. Forexample,astudentdecidestocontinuetakingJAMBexaminationuntilhepasses.X inthis casedenotenumberof timeshetakestheexaminationbeforethefirstsuccess.

Theprobabilitythatthefirstx–1trailsarefailuresandthexthtrailisasuccessisgiven by$(1-p)^{x=1}$p.

Toseethis,therequiredprobabilityis

P(FFF…FS)=P(F)P(F)…P(F)P(S)=(1–p)(1–p)(1–p)…(1–p)P=$(1–p)^{x-1}$p.

Theprobabilitythatxtrailsareneededforthefirstsuccessisthesameastheprobability thatthefirstx–1trailsarefailuresandthexthisasuccess.Thus,

$$P(X=x)=\begin{cases} P(1-p)^{x-1}, & x=1,2\ldots \\ 0 & \text{otherwise} \end{cases}$$

WhereXisthenumberoftrialsbeforethefirstsuccessandifwedefineYasthenumber of failuresprecedingthefirstsuccess,wehave

$$P(Y=y)=\begin{cases} P(1-p)^{y} & y=0,1,2\ldots \\ 0 & \text{otherwise} \end{cases}$$

Definition3.6

(thesuccessfultrialisincludedinthecount).Toseethat(f(x)isaprobability densityfunction,allthatneeded tobecheckedisthat

$(1-p)^{x=1}$=1= P[1+ (1-p)$\sum_{x=1}^{\infty}$(1-p)^2+…]

Fromgeometricseries,

$$\sum_{x=1}^{\infty} (1-p)^{x=1}=\frac{1}{1-(1-p)}=\frac{1}{p}.$$

Example3.11

Afairistosseduntilaheadappears(a)Whatistheprobabilitythatthreetossesare needed?(b)Whatistheprobability thatatmostthreetossesareneeded?

Solution

LetXdenotethenumberoftossesuntilasuccess(ahead)occurs.Sincethecoinisfair,

. $p=\frac{1}{2}$

P(X= 3) =$(1–p)^{3-1}$ p $\left(\frac{1}{2}\right)^2\frac{1}{2}=\left(\frac{1}{2}\right)^3$

P(atmostthreetossesareneeded)=P(X ≤3)

=P(X =1) +P(X =2)+ P(X =3)

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{7}{8}.$$

A fairdieisrolleduntilasixappears.

Whatistheprobabilitythat(i) atmost4rollsareneeded.(ii)atleast3rolls areneeded?

Solution

(i) P(atmost4rollsareneeded)=P(X ≤4)

=P(X =1) +P(X =2)+ P(X =3) +P(X=4)

X isageometricrandomvariablewithP= $\frac{1}{6}$

$$P(X = X) = \left|\left(\frac{5}{6}\right)\right|^{X1} \frac{1}{6}$$

$$P(X = 1) = \frac{1}{6}, P(X = 2) = \frac{5}{6}, \frac{1}{6}, P(X\ 3) \left|\left(\frac{5}{6}\right)\right|^2 . \frac{1}{6}$$

$$P(X = 4) = \left(\frac{5}{6}\right)^3 . \frac{1}{6}$$

Thus

$$P(X \ \leq 4) = \frac{1}{6}\frac{5}{6} . \frac{1}{6}\left(\frac{5}{6}\right)^2 . \frac{1}{6}\left(\frac{5}{6}\right)^3 . \frac{1}{6}$$

$$\frac{1}{6}\sum_{X1}^{4}\left(\frac{5}{6}\right)^{x\ 1} \frac{1}{6} . \frac{1\ (5/6)^4}{1\ \ 5/6}\ 1\ (5/6)^4$$

(ii) P(atleast3rollsareneeded)= P(X≥3) =P(X= 3) +P(X =4) +…

$$P(X \ \geq 3) = \sum_{x\ 3} f(x) \sum_{x\ 3}\left(\frac{5}{6}\right)^{x\ 1} \frac{1}{6}\frac{1}{6}\sum_{x\ 3}\left(\frac{5}{6}\right)^{x\ 1} \frac{1}{6} . \frac{1\ (5/6)^2}{1\ \ \ 5/6}\left(\frac{5}{6}\right)^2 \frac{25}{36}$$

$$P(X \ \geq 3) = 1 - \{P(X = 1) + P(X = 2)\} = 1 - \frac{11}{36}\frac{25}{36}$$

Example3.13

The probability that a certain test yields a “positive” reaction is 0.6. what isthe probabilitythatnotmorethan4negativereactionsoccurbeforethefirst positiveone?

Solution

LetXdenotethenumber of negativereactionsbeforethefirstpositiveone, then

$P(X= x) =(0.4)^{X}(0.6);x=0,1,2,\ldots$

Thus,

$$P(X \leq 4) = \sum_{0}^{4} P(X \quad x) \quad \sum_{x \, 0}^{4} (0.4)^{X}(0.6) \quad (0.6)\sum_{x \, 0}^{4}(0.4)^{x} \quad (1 \quad (0.4)^{5}) \quad 0.9898$$

Thegeometricrandomvariablehasaninterestingpropertywhichissummarizedinthe followingtheorem.

Theorem:3.2
SupposethatXisageometricrandomvariable.Thenforanygiventwopositiveintegers s andt,

$P(X> s +t|X>s)= P(X >t)$.

Proof:

$$P(X> s +t|X>s) = \frac{P(Xs \quad tandX \quad s)P(X \quad s \quad t}{P(Xs) \qquad (X \quad s)}$$

$P(X= x) =p(1- p)^{x-1},x= 1,2,3,\ldots$

$$P(X > s + t) = P \sum_{x \, s \, t \, 1} (1 \, p)^{x \, 1} \quad P \, \frac{(1 \;\, p)^{s \;\, t}}{1 \, (1 \;\, p)} (1 \;\, p)^{s \;\, t}$$

$$P(X > s + t) = P \sum_{s \;\, 1} (1 \, p)^{x \, 1} \frac{P(1 \;\, p)^{s}}{1 \, (1 \;\, p)} (1 \;\, p)^{s}$$

Thus,

$$P(X > s + t| \, X > s) = \frac{(1 \;\, p)^{s \, t}}{(1 \;\, p)^{s}} (1 \;\, p)^{t}$$

$$P(X > t) = \sum_{t\ 1} p(1\ \ p)^{x\ 1} \frac{P(1\ \ p)^t}{p} (1\ \ p)^t$$

Hence,

P(X> s +t|X>s) =P(X >t).

Theabovetheoremstatesthatifasuccesshasnotoccurredduringthefirsts repetitionsof Bernoulitrails,thentheprobabilitythatitwillnotoccurduringthenexttrepetitionsis thesameastheprobabilitythatit willnotoccurduringthefirsttrepetitionsof Bernoulli trails.Thereforethedistributionissaidtohave“nomemory”.

SELF-ASSESSMENT EXERCISE3

1. Onacertainroadtheprobabilityofanaccidentonany dayis0.05(assumingnot more thanone accident can occur on any day). Assuming independence of accidents fromdayto dayon thisroad, what isthe probability that the first accidentof theyearoccurs in themonthof March.
2. Afairdieisrolleduntilasixappears.Calculatetheprobabilitythathehasto throwthediemorethanthreetimesbeforehegetsasix.
3. Let X be a geometric random variable with p = 0.2. Calculate the followingprobabilities. (i) P(3 < x ≤6) (ii) P(2≤X≤4) (iii) P(X ≤2).
4. Prove that $\sum_{x=1}^{\infty} xp(1-p)^{x=1} = \frac{1}{p}.$
5. Suppose a Bernoulli trail with probability p ofsuccess is continued until rth successoccurs.LetXbethenumberofindependenttrailsneededinordertohave r successes.Showthat

$$P(X=K)=\binom{k-1}{r-1}p^r(1-p)^{k-r}, k=r,r+1,...$$

3.2PascalorNegativeBinomialRandomvariable

AprobabilitydistributioncloselyrelatedtothegeometricdistributionisthePascalor negativebinomialdistribution.
SupposethatindependentrepetitionofBernoullitrailneededtohave“success”occurs exactlyr times.

X=xifandonly ifsuccessoccursonthexthtrailandsuccessoccursexactly(r– 1)times

inthepreviousx-1trails.Theprobabilityof this eventisdeterminedasfollows: Theprobability of r–1successesinx–1trailsisgivenbythebinomialformula

$^{X=1}C_{r-1}p^{r-1}(1-p)^{x=1-(r-1)=x-1}C_{r-1}p^{r-1}(1-p)^{x-r}$

Therefore,theprobabilityof thiseventis

$P.^{x-1}C_{r-1}p^{r-1}(11-p)^{x-r+}$

hence,

$f(x) = P(X=x) = {}^{x-1}C_{r-1}p^{r}(1-p)^{x-r}, x=r,r+1,\ldots$ (3.4)

Similarly,ifweletY be thenumberof failuresbeforetherthsuccesswehave

$P(Y=y) = p. ({}^{y+r-1}C_{y}p^{r-1}(1-p)^{y}$

$F_Y(y) = {}^{y+r-1}C_{y}{}^{r}(1-p)^{y}, y=0,1,2,\ldots$

Definition3.7

A randomvariablehavingitsprobabilitydensityfunctiongivenby(3.4)or(3.5) issaidto gaveaNegativeBinomialor Pascaldistribution.

Example3.14

Show that

$$\sum_{x=t}^{\infty} {}^{x-1}C_{r-1}P^{r}(1-P)^{x-r} = 1.$$

Solution

ByBinomialtheorem,

$$P^{-n} = (1-(1-p)^{-n} = \left\{1 + n(1-p) + \frac{n(n+1)}{2!}(1-P)^2 + \ldots\right\}$$

$$\sum_{X=r}^{\infty} {}^{x-1}C_{r-1}P^{r}(1-P)^{x-r} = P^{r}(1-p)^{-r}\sum_{X=r}^{\infty} {}^{(x-1)}C_{r-1}(1-P)^{x}$$

$$= p^{r}(1-p)^{-r}\left\{(1-p)^{r} + r(1-p)^{r+1} + \frac{r(r+1)}{2!}(1-p)^{r+2} + \cdots\right\}$$

$$= p^{r}\left\{1 + r(1-p) + \frac{r(r+1)}{2}(1-p)^2 + \cdots\right\} = p^{r}p^{-r} = 1.$$

Example3.15

A fairdieisrolleduntiltwosixesoccurs,findtheprobabilitythat

(i) exactly5tossesareneeded

(ii) atmost5tossesareneeded.

Solution

LetXbethenumberoftossesneededtogettwosixes.XisaPascalrandomvariable withpdf: f(x) =$^{x-1}C_1p^2(1-p)^{x-2}$,x=2,3,..
Wherep=½,r =1.

$f(x) \quad - (x-1)\left(\frac{1}{2}\right)^x$

(i) $P(X\ 5)0.125 \quad = \quad - 4\left(\frac{1}{2}\right)^5 =$

(ii) $P(X\ 5)\sum_{x-2}^{5} f(x) P(X\ 2) 3) 4) P(X = 5) - \quad - \quad + P(X - \quad + P(X - +$

$P(X=2) = \left(\frac{1}{2}\right)^2, P(X=3) = 2\left(\frac{1}{2}\right)^3, P(X=4) = 3\left(\frac{1}{2}\right)^4$

3.3TheHypergeometricRandomVariable

Supposethatwehaveaboxcontainingnitemsofwhichn1 aredefectiveandn–n1 are non-defective.Supposethatwechooseatrandomkitemsfrom theboxwithout replacement.LetX denotethenumberofdefectiveitemsin thekitemsselected.Then

$$P(X=x) = \frac{^{n_1}C_x \; ^{n-n_1}C_{k-x}}{^{n}C_x}, \quad x = 0, 1, 2, 3 \ldots k \qquad (3.4)$$

ThereaderwillnoticethatX=xifandonlyifweobtainxdefectiveitemsfromn$_1$defectiveitemsinthebox($^{n}{}_1C_x$ ways of doingthis).

Anyrandomvariablehavingitsprobabilitydensityfunctionasgivenby(3.5)iscalledhypergeometricrandomvariableandissaidtohavehypergeometricdistribution.

3.4 ThePoissonRandomVariable

Definition:*RareEvent*

An eventissaidtobe rareiftheprobabilitypofobservingtheeventisvery small. ConsidernrepeatedBernoullitrails,wherenisverylargeandPverysmall,letXbebethenumberof successesin ntrials.Then

P(X= x) =$^{n}C_xp^x(1-p)^{n-x}$.

Settingλ=np,wehave

$$P(X-x) = {}^{n}C_x\left(\frac{\lambda}{n}\right)^{\lambda}\left(1-\frac{\lambda}{n}\right)^{n-x} - \frac{n(n-1)\ldots(n-x+1)}{x1}\cdot\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$
$$= \frac{n}{n}\cdot\frac{n-1}{n}\ldots\frac{n-x+1}{n}\frac{\lambda^x}{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

Now asn ∞.

$$1. \quad \left(1-\frac{\lambda}{n}\right)^{n} \longrightarrow e^{-2}$$

2. $\left(1-\frac{\lambda}{n}\right)^{-x} \longrightarrow 1$

3. $\frac{n-X+1}{n} \longrightarrow 1$

(3.5)givesanapproximationtothebinomialdistributionwithλ=np,whennislargeand pissmall,wheree= 2.71828isthebaseof naturallogarithms.

Definition:
A random variable X is called a Poisson random variable if its probability density functionisgivenby

$$f(x) = \begin{cases} \frac{e^{-\lambda t}(\lambda t)^x}{x!} & x = 0, 1, 2, \ldots \quad (\lambda > 0) \\ 0 & \text{otherwise} \end{cases} \qquad (3.8)$$

Xrepresentsthetotalnumber of eventswhichhaveoccurreduptotimet.When t=1then f(x)correspondstotheprobability density functionofnumberofeventsinaunitinterval. For examples,thenumberof callsthatcomeintoatelephoneexchangein aunittime interval,thenumber of vehiclespassingthrough a designatedpointina unittimeinterval. Inordertomotivateourdiscussion,letus considerthefollowingexamples.

Example3.16
Supposeararediseaseoccursin2percentofalargepopulation.Arandomsampleof 10,000peoplearechosenatrandomfrom thispopulationandtestedforthedisease. Calculatetheprobabilitythatatleasttwopeoplehavetheratedisease.

Solution
Theprobabilitypofhavingthediseaseis0.02andn=10,000.LetXbethenumberofpeoplehavi ngthedisease.Xisabinomialrandomvariablewithparametersn=10,000 andp=0.02.weshallapplytheresult(3.5) sincenislargeandpvery small.

Hence,

$Np = 0.02 \times 10{,}000 = 20_{,,}$

Thus,

$P(X = x) = \frac{e^{-200}(200)^X}{x}$

$P(X =) = e^{-200}$

$P(X = 1) = 200\, e^{-200}$

Hence

$$P(X \geq 2) = 1 - P(X < 2) = 1 - e^{-200}(1 + 200) = 1 - 201\, e^{-200}.$$

Example3.17

On a given road, an average of five accidents occur every month. Calculate theprobabilitythatoverayearperiodtherewillbe(i) noaccident(ii)atmost2accidents.

Solution

In this case, $\lambda = 5$, $t = 12$. From (3.8) we have

$$f(x) = \frac{e^{-\lambda t}(\lambda t)^X}{X!}$$

$$f(0) = e^{-\lambda t} = e^{-60}$$

$$f(1) = \frac{e^{-\lambda t}(\lambda t)^1}{1!} = 60e^{-60}$$

$$f(2) = \frac{e^{-\lambda t}(\lambda t)^2}{2} = \frac{60^2}{2}e^{-60}$$

thus,

$$P(\text{no accident}) = f(0) = e^{-60}$$

$$P(\text{at most two accidents}) = f(0) + f(1) + f(2) = e^{-60} + 60\,e^{-60} + \frac{60^2}{2}e^{-60}$$

$=e^{-60}(1+60+1800)=1861e^{-60}$.

Tables for the Poisson distribution are available and brief tabulation is given in theAppendix.

SELF-ASSESSMENT EXERCISES3.4

1. If 3% of the items manufactured in a factory are defective. Compute theprobabilitythatinasimpleof100items. (i)2itemswillbedefective (ii)atleast2itemswillbedefective.UsePoissonapproximationto theBinomial.
2. UsethePoissonapproximationtocalculatetheprobabilitythatatleasttwosixes areobtainedwhensixdicearerolledonce.
3. ThetelephoneswitchboardofaUniversityhasanaverageoftwoincomingcalls perminutes.Calculatetheprobabilitythat,overathree-minuteinterval,therewill be(i) noincomingcalls (ii)exactlyoneincomingcall. (iii)atmosttwoincoming calls.
4. Let$P_r$bethenegativebinomialpdfwithparametersr andP.provethat $\frac{P_r(k)}{P_r(k-1)} = 1 + \frac{1}{k}\left[(r-1)\frac{a}{p} - k\right]$ $k = 1, 2, \ldots,$ where$P_r(k)=P(X=k)$.
5. Provethat $\sum_{x=0}^{n}\frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{n!}\int_{\lambda}^{\infty} e^{-y}y^n\,dy$. Hint:Useintegrationbyparts.
6. Afairdieistossed20times.Letasuccessontheithtosscorrespondstoatleasta fiveappears.Whatistheprobabilitythat
   (i) exactly10failurepriortothefirstsuccess?
   (ii) exactly10failures priortothefifthsuccess?
   (iii) exactly8failureand3successesforthefirstII throws.
7. if(i) $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ show that $(x+1)f(x+1)=\lambda f(x)$.
   (ii)$f(x)={}^nC_x p^x(1-p)$,show that $(1-p)(x+1)f(x+1) =p(n-x)f(x)$.
   In eachcase, show that f(x) increases monotonically and then decreasesmonotonicallyasxincreases.Findxthatmaximizesf(x).
8. Suppose that thenumber of accidents occurring on ahighway each day isa Poisson randomvariablewithparameter $\lambda = 2$.

(a) Whatistheprobabilitythatatleasttwoaccidentsoccurinaday?
(b) whatistheprobabilitythatatleastthreeaccidentsoccurinadaygiventhat atleastoneaccidentoccursinaday?

9. If$n\geq8$andinsbinomialdistributiontheprobabilityof7successesinntrailsis equaltotheprobabilityof8successesinn trail,findtheprobabilityofsuccesson any giventrail.

10. LetXbearandom variablewithmoment generatingfunction$M_X(t)$.If R(t)= In'' $M_X(t)$forallt.FindR''(0).

11. Ifafairdieisrolledrepeatedly,findtheprobabilitythatthefirst"six"willappear onanodd-numberedroll.

12. $f(x) = \begin{cases} p, x = -1 \\ (1-p)^2p^x, X = 0,1,2 \end{cases}$

(i) Show that f(x) is a pdf

4.0 CONCLUSION

Different ways of calculating probability of different distribution using various random variables especially Pascal random variable, Poisson random variable.

5.0 SUMMARY

It will be noticed that any random variable having its probability density function as $F_y(y) = {}^{y+r-1}C_y{}^r(1-p)^y$, y=0,1,2,… Other topics covered here include Pascal or negative binomial random variable, geometric random variable.

6.0 TUTOR-MARKED ASSIGNMENT

1. DescribethePoissondistribution,statingclearlythemeaningsofthesymbolsused. Show thatthevarianceof thedistributionis equaltoitsmean.

2. Show that$f(x)={}^{x-1}C_{n-1}P^n(1-p)^{x-n}$,x,x=n,n+1,…isapdf.

   Note $P^{-n}=(1-(1-p))^{-n}=$

3. Showthatf(x)= x= 1,2,…,0<p<1is apdf.Findtheprobability generatingfunctionofX.

4. Solvetheproblemof exercise3.2,No7.

5. Let Xbea geometric randomvariable withp= 0.2.Calculate thefollowingprobabilities.(i)$P(3<x\leq6)$ (ii)$P(2\leq X\leq4)$ (iii)$P(X\leq2)$.

that 6. Prove $\sum_{x\ 1} xp(1\quad p)^{x\ 1}\quad \frac{1}{p} . \frac{}{(1-p)^X} \left\{1 + n(1-p) + \frac{n(n+1)}{2!}(1-p)^2 + \cdots\right\}$

7.0 REFERENCE/FURTHER READING

Harry Frank & Steven C. Althoen (1995). Statistics: Concepts and Applications. Cambridge University Press.

UNIT 5 EXPECTATION OF DISCRETE RANDOM VARIABLE

CONTENT

1.0 Introduction
2.0 Objectives
3.0 Main Content
3.1 Expectation of discrete random variable
3.2 Expectation of a Binomial random variable
3.3 Properties of Expectation
3.4 Properties of variance
3.5 The variance of a geometric random variable
3.6 The variance of a Poisson random variable
3.7 Probability generating function
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Reading

1.0 INTRODUCTION

Inthisunit,we introducetheconceptofthe meanvalueofarandom variable.Itis closelyrelatedto thenotionofweightedaverage.Themomentandprobabilitygenerating functionsof arandomvariablearealsointroduced.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- State the mean value of a random variable
- Calculate the mathematical expectation of any variable
- State properties of expectation and variance
- State the probability generating function of random variables(non-negative)

3.0 MAIN CONTENT

3.1 ExpectationOfDiscreteRandomVariable

LetXbearandomvariablehavingpossiblevalues$x_1,x_2,\ldots,x_n$,andlettheexperiment onXbeperformedntimes.ForexampleletXbetheoutcomeofrollingadie.Thereare sixpossiblevalues$x_1=1,x_2=2,x_3=3,x_4=4.x_5=5andx_6=6$.Supposethedieis rolledntimes.Thesuccessiverollsconstituteindependentrepetitionsof thesame experiment.Let$X_1,X_2,\ldots,X_n$denotetheoutcomesoftheexperimentofrollingadientimes(thatis$X_i$denotetheoutcomeofthei^{th}toss). Then

$$\bar{X} = \frac{X_2 + X_2 + \cdots + X_n}{2}$$

Istheaverageofthenumbersthatappeared.Letf_i denotethenumberoftimesx_i occur. Thenwehave

$$+ \quad \frac{1}{n}(X_1 + X_2 \quad \cdots + X_n) - \frac{1}{n}\sum_{i=1}^{k} x_i f - \sum_{i=1}^{k} x_i \frac{f_1}{n}$$

We know that

$\lim_{n\to\infty} \frac{f_i}{n} = P(X = x_i) = f(x_i)$. *When* $\frac{f_i}{n}$ *is replaced by* $f(x_i)$, *the sum* $\sum x_i f(x_i)$

iscalledthe expectationoftherandom variableX. thus theexpectedvalueofarandom variableX isthelong-runtheoreticalaveragevalueof X.

Definition1

MathematicalExpectation.LetXbearandomvariablewithprobabilitydensityfunction asfollows:

Valueof X,x	x_1	x_2	x_3	….	x_k
Probability	$f(x_1)$	$f(x_2)$	$f(x_3)$		$f(x_k)$

ThemathematicalexpectationofX,denotedbyE(X),isdefinedto be

$E(X)= x_1 f(x_2)+ x_2 f(x_2)+\ldots+x_k f(x_k)$

$E(X)=\sum_{i=1}^{k} x_i f(x_i)$.

E(X)isweightedaverageofpossiblevaluesofx,theweightattachedtothevaluexiisits probability$f(x_i)$.

E(X)is alsocalledthemeanofX or ***thepopulationmean.***

Wemay expresstheresultof(1) inwords.

Tocomputetheexpectedvalueormeanofarandomvariable,multiplyeachpossible valueof thevariablebyitsprobabilityandadd theseproducts.

Examples

1. Afaircoinistossedthreetimes.LetXdenotethenumberofheadsobtained.Find themathematicalexpectationof X.

Solution

Theprobability densityfunctionofX is asfollows.

X	0	1	2	3

f(x)	1	3	3	1

Hence,
E(X)= (1×1/8)+(1×3/8)+(2×3/8)+(3×1/8)
=0+ 3/8 +6/8 +3/8 =12/8=1.5

2.WhatatethemathematicalexpectationsofthevariablesX,YandZasdefinedin example3.1.1 of chapter3.

Solution

Theprobability densityfunctionofX is

X	2	3	4	5	6	7	8	9	10	11	12
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

E(X)= (2×1/36)+(3×2/36)+(4× 3/36)+ (5×4/36)+(6×5/36)+(7× 6/36)+(8×5/36)+(9×4/36)+(10×3/36)+(11×2/36)+(12×1/36)=7.0

Thep.d.fofY is

Y	1	2	3	4	5	6
g(y)	1/36	3/36	5/36	7/36	9/36	11/36

E(Y)= (1×1/36)+(2×3/36)+(3×5/36)+(4× 7/36)+ (5×9/36)+(6×11/36)=4.47

Thep.d.fofZis

Z	1	2	3	4	5	6
h(Z)	11/36	9/36	7/36	5/36	3/36	1/36

E(Z)=(1×11/36)+(2×9/36)+(3× 7/36)+(4×5/36)+(5×3/36)+(6× 1/36)=2.53.

SELF-ASSESSMENT EXERCISES1

1. Supposeaboxcontains10ballsofwhich4areredand6areblack.Arandom sampleof size3isselected.LetX denotethenumberof redballsselected.Find (E(X)if
   (i) Samplingiswithreplacement.
   (ii) Samplingiswithoutreplacement.
2. A boxcontains6ballslabeled1,2,3,4,5,6. Twoballsaredrawnatrandom one afterthe other.LetXdenotethelargerofthetwonumbersontheballsselected. ComputeE(X)if
   (i) Samplingis with replacement,

(ii) Samplingis without replacement.

3. Aboxcontains3ballsand2whiteballs.Aballiswithoutreplacementoneafter theotheruntilawhiteballis drawn.FindtheExpectednumber ofdrawsrequired.

3.3 ExpectationofaBinomialRandomVariable

Theprobability densityfunctionofabinomialvariableX is defined by $f(x) = {}^{n}C_{x}p^{x}(1-p)^{n-x}$;x= 0,1,2, …,n.
themeanorexpectedvalueofX is

$$E(X)=\sum_{x=0}^{n} x\, p^{x}(1 \quad {}^{n}C_{x}p^{x}(1-p)^{n-x} - \sum_{x=0}^{n} x \frac{n!}{x!(n-x)} \quad -p)^{n-x}$$

andsincetheterm of x=0iszero, E(X)=

$$\sum_{x=1}^{n} \frac{n}{(x-1)!(n-X)!} p^{X}(1-p)^{n-x} = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-X)!} p^{X-1}(1-p)^{n-x}$$

Using thebinomialtheorem,wehave

$$\sum_{x=1}^{n} \frac{(n-1)}{(x-1)!\,(n-X)!} p^{X-1}(1-p)^{n-x} = \sum_{x=0}^{n-1} {}^{n-1}C_{x}p^{x-1}(1-p)^{n-x}$$

sincep+1–p=1.Hence,
E(X)= np.

Example 3.3
A fairdieisrolled12times,whatistheexpectednumberof sixappear?

Solution
LetXbethenumberofsixesthatappear.Xisabinomialrandomvariablewithn=12 andP= 1/6(probabilityofasix).Hence
E(X)= np=12x 1/6=2.
Thatis,weexpectedtoget2sixeswhenadieisrolled 12times.

Definition2
LetXbeadiscreterandomvariablehavingprobabilitydensityfunctionf(x).

$\sum_{i=1}^{\infty} x_i f(x_i) < \infty$ thenwesaythatXhasfiniteexpectationotherwisewesaythatXdoes nothavefiniteexpectation.

Examples

3.4TheRandomVariablesX hasProbability

X	-2	-1	0	3	5

f(x)	¼	1/8	1/8	¼	¼

Computetheexpectedvalueof thefollowingrandom variables.

(i) X, (ii)3X, (iii)X +5, (iv)X^2

Solution

(ii) $Ex = \left(-2x\frac{1}{4}\right) + \left(-1x\frac{1}{8}\right) + \left(0x\frac{1}{8}\right) + \left(3x\frac{1}{4}\right) + \left(5x\frac{1}{4}\right)$

$= -2/4 - 1/8 + 3/4 + 5/4 = 11/8 = 1\frac{3}{8}$

(iii) Thepossiblevaluesof3x are-6,-3,0,9,15.
Theprobability densityfunctionof3xis

3x	-6	-3	0	9	15
f(x)	¼	1/8	1/8	¼	¼

E(3X)=(-6× ¼) +(-3×1/8)+(0×1/8)+(9×1/4) + (5×¼)
=-6/4– 3/8+ 9/4+15/4= 33/8

Notethat

P(3X =-6)= P(X =-2_,

Andso onand

E(3X)=33/8=3×11/8=3E(X)

(iii)

x+5	3	4	5	8	10
f(x)	¼	1/8	1/8	¼	¼

NotethatP(X +5)=3=P(X= 3–5)= P(X =-2)
E(X+ 5) =(3× ¼) +(4×1/8)+ (5×1/8)+(8× ¼)+ (10×1/4)
=¾ + 4/8+5/8+8/4+10/4+51/8=6 = 5+1 $\frac{3}{8}$=5+E(X)$\frac{3}{8}$

(iv)

X^2	4	1	0	9	25
f(x)	¼	1/8	1/8	¼	¼

E(X^2)=(4× ¼) +(1×1/8)+(0×1/8)+(9× ¼) +(25×¼)

$=1+ 1/8+9/4+25/4= 77/8= 9\frac{5}{8}$

Note that $E(X^2) \neq [E(X)]^2$.

Definition3
LetXbearandomvariablewhoseprobabilitydensityfunctionisgivenby

X	x_1	x_2	…………	x_k
f(x)	$f(x_1)$	$f(x_2)$		$f(x_k)$

Let $\psi(X)$ be a function of X. then the mean or expected value of the new random $\psi(X)$ is given by

$E[\psi(X)]= \psi(x_1)f(x_1)+ \psi(x_2)f(x_2) +\ldots+\psi(x_k)f(x_k)$

Thatis

$E[\psi(X)]= \sum_{i=1}^{k}\psi(x_i)f(x_i)$

Theorem1
LetXbearandomvariableandlet $\psi(X)= aX+b$ where a,b areconstants,then

$E(\psi(X))= aE(X)+b$

Proof:
Suppose theprobabilitydensityfunctionof Xis
$\{x_i,f(X_i), I= 1,2, \ldots,k)$

Fromthe abovedefinitionwehave
$E(aX+b)= (ax_1+b) f(x_1)+ (ax_2+ b) f(x_2)+\ldots +(ax_k+ b) f(x_k)+bf(x_1) +\ldots+bf(x_k)$
$=a[x_1f(x_1) + x_2f(x_2)+\ldots x_kf(x_k)]+b[f(x_1)+ f(x_2)+\ldots+$
$f(x_k)]=\sum_{i=1}^{k} x_i f(x_i)\ a E(X)\ b \sum_{i=1}^{k} f(x_i) = \quad + b$
since $\{E(X)=\sum xf(x), \sum f(x_i) 1\}$

PropertiesofExcitations
1. If c isaconstantand $P(X =c)= 1$, then $E(X)=c$.
2. If a and b are constants and X has a finite expectation, then aXhas finite expectationand
   (i) $E(aX)=aE(X)+b$.
   (ii) $E(aX+b) =aE(X)+b$.
3. If XandY aretwo randomvariableshavingfiniteexpectations,then
   (i) X +Y hasrandomfiniteexpectationand $E(X+ Y)= E(X)+ E(Y)$
   (ii) $E(X)\geq E(Y)$ if $P(X \geq Y)= 1$.

(iii) DEXD≤EDXD.

4. Aboundedrandomvariablehasafiniteexpectation.ThatisifP(X≤M)=1,then X hasafiniteexpectationandDEXD≤M.

Definition4

Variable.LetX bea randomvariablewithmeanE(X)=μ. Thevarianceof X,denotedbyVar(X)isdefined by

$Ver(X) = E\{X-\mu)^2\} = \sum(x_i-\mu)^2 f(x_i)$ –

A generalformulathatis usuallysimplefor computingthevariableisgivenbelow

$Var(X)=E\{(X-\mu)^2\}=E\{X^2-2X\mu+\mu^2\}$

$=E(X^2)-E(2\mu X)+E(\mu^2)=E(X^2)-2\mu E(X)+\mu^2$(property2)

$=E(X^2)-2\mu^2+\mu^2=E(X^2)-\mu^2$.

Thus,thecomputing definitionofvarianceis

$Var(X)=E(X^2)-[E(X)]^2$(7)

ThevarianceofXisinterpretedasanumericalmeasureofspreadordispersionaboutits mean.

Example 3.5

1. Findthevarianceofarandomvariablehavingthefollowingprobabilitydensityfunction.

X	1	2	3	4	5	6
f(x)	1/6	1/6	1/6	1/6	1/6	1/6

E(X)= (1×1/6)+(2×1/6)+(3×1/6)+(4×1/6)+ (5×1/6)+(6×1/6).
=1/6+2/6+3/6+4/6+ 5/6+6/6=21/6

$E(X^2)= (1^2\times1/6)+(2^2\times1/6)+(3^2\times 1/6)+(4^2\times 1/6)+(5^2\times 1/6)+(6^2\times1/6)$
=1/6+4/6+ 9/16+16/6+25/6+36/6= 91/6.

Hencefrom (7)wehave

$Var(X)=91/6-(21/6)^2= 105/36= 2(11/12)$.

Example 3.6

FindthevarianceofaBernoullirandomvariablewithparameterp.thatisXhas thefollowingp.d.f.

X	0	1
f(x)	1–p	P

E(X)–0×(1-p) +(1×P) =p
$E(X^2)= 0^2\times(1-p) +(1^2\times p) =p.$

Hence,

$Var(X)=p-p^2= p(1–p)$

PropertiesofVariance

1. If c isaconstantandP(X) =c)=1, thenVar(X) =0.
2. If aandbarecontents,then
   (i) $Var(aX)= a^2Var(X)$
   (ii) $Var(aX+ b)=a^2Var(X)$
   (iii) Var(X)≥0

Theproofofproperty(1)isverytrivialandthisisleftasanexercisetothereader, weshallnow giveaproof of (ii).From(7),
$Var(aX)= E[aX^2]–[E(aX)]^2= Ea^2X^2–[aE(X)]^2$
$=a^2E(X^2)–a^2[E(X)]^2= a^2\{(E(X^2)]^2\}=a^2Var(X).$

Definition5

Standarddeviation.LetX bearandomvariablewithmeanμ.

Thestandard deviationisthepositivesquarerootofthevariance,andisgivenby

$$\sqrt{Var(X)}= \sqrt{E(X^2)- (E(X))^2}$$

Thevarianceof thebinomialrandom variable.

TocalculatethevariableofX,weneedE(X)andE(X^2).

$Var(X)=(E(X))^2.$

From(3)wehave

E(X)=np.

$E[X(X-1)] = \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x(1-p)^{n-x} = \sum_{x=0}^{n} x(x-1)\frac{n}{x!(n-x)!} p^x(1-p)^{n-x}$

(w2henx=1 or 1the expressioniszero)

$= \sum_{x=2}^{n} \frac{n}{(x-2)!(n-x)} p^x(1-p)^{n-x}$

$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)} p^{x-2}(1-p)^{n-x}$

Let $y = x-2 = n(n-1)p^2 \sum_{y=0}^{n-2} \binom{n-2}{y} p^y(1-p)^{n-2-y} = n(n-1)p^2[P+(1-p)]^{n-1}$

Hence

$E(X(X-1)) = n(n-1)p^2$

$E(X(X-1)) = E(X^2) - E(X) = n(n-1)p^2$

$E(X^2) = n(n-1)p^2 + E(X) = n(n-1)p^2 + np.$

Thus,

Var(X)=N(n– 1) p^2+np–n^2p^2=n^2p^2–np^2+ np–n^2p^2=np(1– p).

3.5 TheVarianceofa GeometricRandomVariable

Theprobability densityfunctionofX isgivenby
f(x) =$p(1-p)^{x-1}$,x=1,2,…
E(X)=$\sum_{x=1}^{\infty} xp(1-p)^{x-1} = \frac{1}{p}$
E(X)= P+2p(1- p)+3p$(1-p)^2$+… =P{1+ 2(1+ p) +3$(1-p)^2$+…}.
FromtheBinomialtheorem
$(1-a)^{-2}$=1+ 2a+3a^2+…DaD<D.
Puttinga=(1– p),weobtain1
E(X)= P.$(1-(1-p))^{-2}$= $\frac{1}{p}$

E[X(X – 1)] =$\sum_{x=2}^{\infty} x(x-1)P(1-p)^{x-1}$

=p[2.1(1-p)+3.2$(1-p)^2$+…+r(r-1)$(1-p)^{r-1}$+…]. Multiplyboth sidesby 1-p,
(1–p) E[X(X-1)]=p(2.1$(1-p)^2$+(3.2$(1-p)^3$+…+r(r-1)$(1-p)^r$+…+]

E(X(X-1)] –(1-p)E[X(X-1)] =p[2(1-p)+ 4$(1-p)^2$+6$(1-p)^3$+…+2r$(1-p)^r$+…]

=2p(1–p)[+ 2(1-p) +3$(1-p)^2$+…]=2p(1-p). $\frac{1}{p^2} = \frac{2(1-p)}{p}$

Hence

E[X(X)]–(1-p) E[X(X– 1)]= $\frac{2(1-p)}{p}$

E[X(X–1)]{1–(1-p)} = $\frac{2(1-p)}{p}$

whichimpliesthat

E[X(X– 1)] = $\frac{2(1-p)}{p^2}$

E(X^2)= $\frac{2(1-p)}{p^2}$+E(X) = $\frac{2(1-p)}{p^2} + \frac{1}{p}$

Thus

Var(X)= $\frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2(1-p)+p-1}{p^2} = \frac{1-p}{p^2}$

3.6 TheVarianceofaPoissonRandomVariable

LetXbeapoissonrandomvariablewithparamentλ.

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=1}^{\infty} e^{-\lambda}\frac{\lambda^{x}}{(x-1)!} \quad \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty}\frac{\lambda^{y}}{y!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda}\lambda^{2}\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} = e^{-\lambda}\lambda^{2}\sum_{y=0}^{\infty}\frac{\lambda^{y}}{y!} = \lambda^{2}$$

$$E(X^2) - E(X) = \lambda^2$$

$$E(X^2) = \lambda^2 + E(X) = \lambda^2 + \lambda$$

Hence,

$$Var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Thisshows that themeanandvarianceof apassionrandomvariablearebothequal.

SELF-ASSESSMENT EXERCISE2

i. ComputethevarianceandstandarddeviationoftherandomvariableXdefinedin exercises4.1.1

ii. A faircoin is tossed twice. Let Xdenote the number of heads that appear. Compute

(i) theexpectedvalueofX

(ii) thevarianceof X

(iii) theexpectedvalueof $\frac{1}{X}$

iii. Calculatethemean,varianceandstandarddeviationoftherandom variablehavingthefollowingprobability densityfunction.

X	-2	1	0	1	2
f(x)	3/10	1/5	1/10	1/5	1/5

iv. LetXbeany random variance,show that$E(AX+B)=aE(X)+B$ Wherea,barecontants.

v. Let$f(x) = \frac{1-p}{2p} p^{|x|}$,$x=\pm1,\pm2,\ldots,0<p< 1$

Show thatf(x)isapdf.FindE(X).

6. LetXbeany random variablehavingfiniteexpectation.Provethat $|EX| \le E|X|$.

7. LetXbearandomvariablesuchthatforsomeconstantK,P(DXD≤K).Provethat

   X hasfiniteexpectationandDEXD≤k.

8. Show that if X and Y are any two dependent random variables with finitevariances,thenVar(X+Y)≠Var(X)+Var(Y).

3.7ProbabilityGeneratingFunction

Inthissectionwe shallintroduceimportantmathematicalconceptsinprobabilitytheory. TherthmomentofarandomvariableXisdefinedbyE(X^r),whereE(X^r)andE(X-μ)^r arecalledtherthmomentofX about0andμrespectively.
From(2)

$E(X^r) = f \sum and\ E(X-\mu)^r \sum (x_i - \mu)^r f(x_i)$

NotethatthesecondmomentofXaboutμisthevarianceofX.anindirectway of calculating expectation is the use of probability generating function or moment generatingfunction.itisamathematicaldevicetosimplifythecalculationsofmoments of X.itsmoregeneralusefulnesswillnotbeapparentuntilwegettochapter6.

Definition6

Theprobabilitygeneratingfunctionofanon-negative,integer-valuedrandomvariableX

is definedby

$E(S^X)=P(S) = \sum_{x=0}^{\infty} S^x f(x);$(8) From(6),weseethat $s < s_0$

$P(S) = E(S^X)$

By differentiatingwithrespecttoswehave
$P'(s) = E(XS^{X-1})$
$P''(s) = E[X(X-1)S^{X-2}]$
$P^{(r)}(s) = E(X(X-1)(X-2)\ldots(X-r+1)s^{X-1})$

Puttings=1 in theabovederivatives,wehave
$P'(1) = E(X)$

$P''(1) = E[X(X-1)] = E(X^2) - E(X)$
$P^{(r)}(1) = E[X(X-1)(X-2)\ldots(X-+1r)]$.

ThusthemeanandvarianceofX canbeobtainedfromP(s) bythefollowingformulas.

$E(X)= P'(1)$

$E(X^2)–E(X)=P''(!)$

$E(X^2)= P''(1)+ E(X)= P''(1)+ P'(1)$

And

$Var(X)=E(X^2)–[E(X)]^2= P''(1)+ P'(1)–[P'(1)]^2$

Wenowillustratetheuseof theseformulaswith thefollowingexamples.

Examples3.7

LetXbeapoissonrandomvariablewitparameterλ.FindthemeanvarianceofX

$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, ...$

Theprobability generatingfunctionisdefinedby

$P(S) =E(S^X)=\sum_{x=0}^{\infty} s^x (x p^X(1–p)^{n-x}= \sum_{x=0}^{\infty} {}^n(^x(sp)^x(1–p)^{n-x}.$

BytheBinomialtheorem

$=[sp+(1–p)]^n$

Thus,

$P(S) =[sp+ (1– p)]^n(12)$

Differentiating,wehave

$P'(s)=np[sp+(1–p)]^{n-1}$

$P''(s)= n(n–1)p^2[sp+1–p]^{n-2}$

$Var(X)=p''(1) +-[p'(1) –[p'(1)]^2=n(n– 1)p^2+np–(np)^2=np(1– p).$

Thetablebelowsummarizessomeof theresultsofthischapter

Random variable	Probability density function	Mean	Variance	Probability generatingfunction	Moment generating function
Bernoulli	$P^x(1-p)^{1-x}$	P	P(1-p)	(sp+1– p)	$pe^t +(1-p)-\infty<t<\infty$

Binomial	${}^nC_xp^x(1-p)^{n-x}$ x=0,1,2,….n	Np	Np(1-p)	$[sp+(1-p)]^n$	$[pe^t+(1-p)]^n$
Geometric	$p(-p)^{x-1}$ x=1,2,…	$\frac{1}{p}$	$\frac{}{p^2}$	$P[1-s(1-p)]^{-1}$	
Poisson		Λ	Λ	$e^{\lambda(s-1)}$	$e^{\lambda(e^t-1)}$
Uniform	$\frac{1}{b}$ X= a, a + 1,…b	$\frac{b}{2}$	12	s^{b+1}	

TheMomentGeneratingFunction

Definition7

Let Xbeadiscrete randomvariable with pdff(x).themoment generating function denoted by$M_X(t)$is definedby

$M_X(t)= E(e^{tX})$

Wheretisany realconstantfor whichtheexpectationexists. (13)

$M_X(t)= \sum_{\forall x} e^{tx} f(x)$

$R^{tX}=1+tX+ \frac{(tX)^2}{2!} +\ldots$

$E(e^{tX})=E(1+tX+ \frac{(tX)^2}{2} +\ldots+ \frac{t^n X^n}{n!}$

Underfairlygeneralconditionsweshallassume thatexpectationofinfinitesumequals

thesumoftheexpectedvaluebutthisistrueingeneralforfinitesum

$1+tE(X)+ \frac{t^2}{2!}E(X^2) + \frac{t^n X^n}{n}\ldots$ Thatis

$M_X(t)= 1+tE(X)+ E(X^2)\frac{t^2}{2} E(X^2) + \ldots + \frac{t^n}{n!} + \cdots$

Differentiatingwithrespectto t, wehave

$M'_X(t) =E(X^2)+\ldots+ \frac{n(n-1)t^{n-2}}{n!}E(X^n) + \cdots$

$M''_X(0)=E(X^{2+})$

hence,

$Var(X)=M_X''(0)-[M_X'(0)^2]$

And

$$M_x(t) = E(e^{tX}),\ M_x^{(r)}(t) = E[X^r e^{tX}] \qquad (15)$$

Putting t = 0, we have

$$M^{(r)}(0) = E(X_r)$$

thatis, $E(X^r)$ istherthderivativeof $M_X(t)$ evaluatedatt=0. (16)

Note:Itis assumedthat $\sum e^{tX} f(x)$ convergesfor allvaluesof t.

ThedomainofM$_X$(t)isallrealnumbertsuchthat $\sum e^{tX} f(x)$ converges.IfM$_X$(t)isdefined for somet$_0$>0,thenitis definedintheinterval$0 \le t \le t_0$.

Examples4.9

LetXbeabinomialrandom variablewithparametersn andp.then

$M_X(t)= E(e^{tX})=\sum_{x=0}^{n} e^{tx}\, {}^nC_x P^x(1-p)^{n-x} = \sum_{x=0}^{n} {}^nC_x(Pe^t)^x(1-p)^{n-x}$

fromthebinomialtheorem

$(a+b)^n = \sum_{x=0}^{n} {}^nC_x a^x b^{n-x} = [pe^t + (1-p)]^n$,

onputtinga= pe^t,b=1- p.

Example4.10

LetXbeaPoisson randomvariablewithparameterλ.Then

$M_X(t)= \sum_{x=0}^{\infty} e^{tX} e^{-\lambda} e^{\lambda et} = e^{\lambda(et-1)} \frac{e^{-\lambda}\lambda^x}{x} = e^{-\lambda} \sum_{x=0} \frac{(\lambda e^t)^x}{x!}$

Differentiating,wehave

$M'_X(0)=\lambda e^t e^{-\lambda(1-et)}$

$M'_x(0)=\lambda e^t e^{-\lambda(1-et)} D_{\lambda=0} = \lambda E(X)$

$M_X''(t)=\lambda e^t e^{-\lambda(1-et)} + \lambda^2 e^{2t} e^{-\lambda(1-et)}$

$M_X''(0)=\lambda+\lambda^2= E(X^2)$

Hence,
$Var(x)=\lambda+\lambda^2-\lambda^2=\lambda.$

Example4.11

LetXbeageometricrandomvariablewith parameterp.them

$$M_x(t) = \sum_{x=1}^{\infty} e^{tX} P(1-p)^{x-1} = \frac{p}{1-p} \sum_{x=1}^{\infty} e^{tX} P(1-p)^{x}$$

$$= \frac{p}{1-p}\sum_{x=1}[e^t(1-p)]^x = \frac{p}{1-p}.\frac{e^t(1-p)}{1-e^t(1-p)} = pe^t[1-e^t(1-p)]^{-1}.$$

Thefourthequalityfollowsfromsuntoinfinityofgeometricseries $\sum_{x=1}^{\infty} a^x$Further applicationsof $M_X(t)$ willbeconsideredlaterinChapter6.

SELF-ASSESSMENT EXERCISE4.3

i. Find theprobability generating function of the geometric distribution with parameterP.hence determinethemeanandvarianceof thedistribution.

ii. LetXbeuniformlydistributedon(0,1,2,…n).Findthemean and varianceofX.

Hint: $\sum_1^n x = \frac{n(x-1)}{2}, \sum_1^n x^2 = \frac{n(n+1)(2n+1)}{6}$

iii. LetXbearandomvariablewithfinitevariance.Provethatforanyrealnumbera, $Var(X)=E[X-a)^2]-[E(X)+a]^2$.

iv. A fairdieistossed50times.LetXbethenumberoftimessixappears.Evaluate

(i)E(X) (ii)$E(X^2)$ (iii)VAR(X)

v. Findtheexpectedvalueandvarianceof therandomvariableXosExample3.1.1.

vi. Show thatifcisaconstant,Var(x+c)= Var(X).

vii. LetXbeuniformlydistributedon(0,1,2,..N).Find P(S),mean andvarianceof X.

viii. LetXbedefinedby

x	1	2	3
f(x)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

FindP(S)andhencethemeanandvarianceofX.

ix. ProvethatifX hasvarianceσ^2, andmeanμ,then

$\sum_{i=1}^{\infty}(x_i a)^2 f(x_i)(a-\mu)^2$everyrealnumbera.

4.0 CONCLUSION

In this unit, emphasis have been laid on the processes of using discrete random variable, properties of variance and expectations and some of the probability generating functions.

5.0 SUMMARY

In this unit, important mathematical concepts in probability theory were discussed. Also, the moment generating function were also generated.

6.0 TUTOR-MARKED ASSIGNMENT

1. DerivetheMGFfor thefollowing:(a)Bernoulli, (b) NegativeBinomial, (c) Binomialanduse itto find themeanandvariance.
2. A randomvariableX hasthepdff(x) definedby

$$f(x) = \begin{cases} c(x+2), & x = 1, 2, 3, 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find(i)c,(ii)momentgeneratingfunctionofX,(iii)useresultof(ii)tofindthe meanandvarianceofX.

3. Provethatfor anyrandomvariable,X,$E(X^2) > [E(X)]^2$.
4. Show that$[E(X-a)^2]$ isminimizedwhena=E(X).
5. A fairdieisrolleduntilallthe6sidesappearedatleastonce.Findtheexpected numberof rollsneeded.
6. Let $f(x) = \begin{cases} p, & x = -1 \\ P^x(1-p)^2, & x = 0,1,2,\dots \end{cases}$

(a)Show thatf(x)isapdf. (b) Find$M_X(t)$,themomentgeneratingfunction of(b)Hence,themeanand varianceof X.

7.0 REFERENCE/FURTHER READING

Harry Frank & Steven C. Althoen (1995). Statistics: Concepts and Applications. Cambridge University Press.