MODULE4

UNIT1 LIMIT THEOREMS

CONTENT

1.0 INTRODUCTION

Themostimportanttheoreticalresultsinprobabilityarelimittheorems.Inthisunit weprove oftenuseful tools in probability,theChebychev'sinequality.Thisinequalityis then usedtodeducethelawof largenumbersfor independentand identicallydistributed randomvariables.Insection8.5thecentrallimittheoremwhichisafundamentalresultin thetheory andapplicationsf probabilitytheoryis given.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Prove the Chebychev's inequality
- Verify inequalities using Chebychev's inequality
- State the law of large numbers of Bernoulli Trails
- Apply approximation theorem for solution of real functions

3.0 MAIN CONTENT

3.1 Chebychev'sInequality

Chebychev;sinequalitygivesanupperboundintermsofvarianceofarandom variableXfor probabilitythatXdeviatesfrom itsmeanby more thankunits.

Theorem 3.1:*Chebychev'sInequality*

LetXbearandomvariable(discreteorcontinuous)withmeanμandvarianceσ2.Then foranypositivenumber kwehave

$$P\{|X-\mu| > k\} \leq \frac{\sigma^2}{k^2}$$

Or equivalently

$$P\{|X-\mu| \geq k\,\sigma\}\} \leq \frac{1}{k^2}.$$

Proof:

LetXbeanon-negativerandom variablesuchthatE(X)=μ<∞.
DefineanotherrandomvariableY asfollows:

$$Y = \begin{cases} 0 & \text{if } X < k \\ k & \text{if } X > k \end{cases}$$

ThisnewvariableYisadiscretevariablehavingtwovalues0,k.Theprobabilitydensity functionofY canbewrittenthus:

y	0	k
P(Y =y)	P(X <k)	P(X >k)

Hence,

E(Y)=OP(X<k) +kP(X >k)

Thatis

E(Y)= kP(X ≥k).

SincethevariableX≥Y forallpossiblevalues,wehave
E(X)≥E(Y)= kP(X≥k),

Thus,

$$P(X \geq k) \leq \frac{E(X)}{k}. \qquad (2)$$

Equation(2)iscalledtheMarkovinequalityandcanbegeneralizedthus:Fora nyj≥0,k>0

$$P\{|W| \geq k\} \leq \frac{E|W|^j}{k^j} \qquad (3)$$

TheproofofChebychev'sinequalityisanimmediateconsequencesof(3)byp uttingj=2 andW =X -μ,weobtain

$$P\{|X-\mu| \geq k\} \leq \frac{E(X-\mu)^2}{k^2} = \frac{\sigma^2}{k^2}$$

Since $(X-\mu)^2 \geq k^2 \Longleftrightarrow |X-\mu| > k$, we have

$$P\{|X-\mu| > k\} = P\{(X-\mu)^2 \geq k^2\} \leq \frac{\sigma^2}{k^2}.$$

Thus

$$P\{|X-\mu| \leq k\} \leq \frac{\sigma^2}{k^2}.$$

Example 3.1

LetXbearandomvariablehavingPoissondistributionwithmeanλ andvarianceσ^2.UseChebychev'sinequalitytoshowthat

(i) $P\{P|X-\lambda| \geq 1\} \leq \lambda$ (ii) $P\left(X > \frac{3\lambda}{2}\right) \leq \frac{4}{\lambda}$.

Solution

Using Chebychev'sinequalitywith$\mu=\lambda$,and$\sigma^2 = \lambda$wehave

(i) $P\{|X-\mu| \geq k\} \leq \frac{\lambda}{k^2}$.

Putting k = 1 we obtain

$$P\{|X-\mu| \geq k\} \leq \lambda.$$

(ii) $Let \frac{\sigma^2}{k^2} = \frac{4}{\lambda} ===> \sigma^2 = \frac{4k^2}{\lambda}$

On substituting for σ^2 we have

$$\lambda^2 = 4k^2,\ k^2 = \left(\frac{\lambda}{2}\right)^2, k = \frac{\lambda}{2}.$$

Thus

$$P\{|X-\mu| \geq \frac{\lambda}{2}\} \leq \frac{4}{\lambda}.$$

Since $|X-\lambda| \geq \frac{\lambda}{2}$ if and only if $X-\lambda < \frac{-\lambda}{2}$ or $X-\lambda > \lambda/2$, we see that

$$P\{|X-\lambda| \geq \frac{\lambda}{2}\} = P\{X-\lambda < -\frac{\lambda}{2}) + P\left(X-\lambda > \frac{\lambda}{2}\right)$$
$$= P(\left(X < \frac{\lambda}{2}\right) + P\left(X > \frac{3\lambda}{2}\right) \leq \frac{4}{\lambda}$$

Since $P\left(X < \frac{\lambda}{2}\right) \geq$, *we have*

$$P\left(X > \frac{3\lambda}{2}\right) \leq \frac{4}{\lambda}.$$

3.2 TheLawofLargeNumbersofBernoulliTrails

Let$X_1,X_2,\ldots,X_n$beindependentandidenticallydistributedBernoullirandom variables andlet$X=X_1+X_2+\ldots+X_n$beBinomialrandomvariables(numberofsuccesses),with parametersn andp.ThemeanandvariableofXarenp andnpqrespectively.

Themeanμgrowsasnincreasesbutthestandarddeviationgrowsonlyas $\sqrt{n}$,

UsingChebychev'sinequality

$$P\{|X-np| > \varepsilon\} \leq \frac{npq}{\varepsilon^2} \leq \frac{n}{\varepsilon^2}.$$

Theorem8.2

LetXbethenumberofsuccessesinnindependentBernoullitrailswithprobabilitypofsuccess.For anyε>0.

$$P\left\{I\frac{X}{n} - PI > \varepsilon\right\} \leq \frac{pq}{n\epsilon^2} \leq \frac{1}{4\epsilon^2 n}$$

and

$$\lim_{n\to\infty} P\left\{I\frac{X}{n} - PI > \varepsilon\right\} = 0$$

Proof:

Applying Chebychev's inequality

$$E\left(\frac{X}{n}\right) = P, Var\left(\frac{X}{n}\right) = \frac{1}{n^2}.npq = \frac{pq}{n}$$

$$\frac{Var\left(\frac{X}{n}\right)}{\epsilon^2} = \frac{pq}{n\epsilon^2} -> 0 \; as\, n -> \infty$$

$$pq = (1-p)p \leq \frac{1}{4} \; for\ all\ p, 0 \leq p < 1$$

Hence

$$P\left\{\left|\frac{X}{n} - p\right| \geq \epsilon\right\} -> 0 \; as\, n -> \infty$$

Thismeansthatforlargenumberwecanbealmostcertainthat $\frac{X}{n}$ willbeveryclosetoprobabilitypofsuccess.Thisshowsthattherelativefrequencyofsuccessi nindependentBernoullitrailsconverges(inaprobabilisticsense)tothe theoreticalprobabilityp of successateachtrail.Thisimportantresultcanbestatedmore preciselyas thelawoflarge numbers.

TheLawofLargeNumbers

3.3 TheWeakLawofLargeNumbers

LetX_1,X_2,…,X_n beindependentandidenticallydistributionrandomvariableswithall $E(X_i^2)<\infty$ andletS_n–X_1+X_2+…+X_h.

Weknow that

$E(X_1)$= $E(X_2)$= …$E(X_n)$,$E(S_n)$= n$E(X_1)$andVar(S_n)=nVarX_1

$$E\left(\frac{S_n}{n}\right) = \frac{1}{n} E(S_n) = \frac{1}{n}.n\, E(X_1) = E(X_1)$$

$$\text{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2} Var(S_n) = \frac{nVarX_1}{n^2} = \frac{1}{n} Var(X_1)$$

Thisshowsthatexpectation of $\frac{S}{n}$ isequaltotheexpectation of$X_i$andthestandarddeviationof $\frac{S_n}{n}$is

$$\sqrt{\left\{\frac{VarX}{n}\right\}} = \frac{StdX_1}{\sqrt{n}}$$

Whichtendstozeroasntendstoinfinity.Thus,thedistributionofS_n/nbecomesmore andmoreconcentratednear$E(X_1)$.
Westatethemoregeneralresultinthefollowingtheorem.

Theorem8.3:A (WLLN)WeakLawof LargeNumbersfor IndependentRandom Variables.Let$X_1,X_2\ldots X_n$ beindependentandidenticallydistributedrandomvariables withfinitemeanμ andσ^2.Let$S_n=X_1-X_2+\ldots+X_n$.thenforδ>0.

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n}-\mu\right|>\delta\right)=0 \qquad (4)$$

Proof:

Applying Chebychev's inequality to $\frac{S_n}{n}$, we have

$$\mu = E\left(\frac{S_n}{n}\right).Var\left(\frac{S_n}{n}\right)=\frac{1}{n^2}Var(S_n)$$

$$P\left\{\left|\frac{1}{k}S_n - E(X)\right|>\delta\right\}\le\frac{2\sigma^2 x}{\sigma^2 n}$$

For atleastoneksatisfyingn≤k≤m}.

Corollary1:

Letf(x)beacontinuousrealfunctionon[0,1].Thenasn->∞uniformlywithrespectto

$$E\left\{f\left(\frac{S_n}{n}\right)\right\}\to f(p).$$

CorollaryII: WeierstressApproximationTheorem.

Letf(X)beacontinuousrealfunctionon[0,1]definedontherealinterval0≤p≤1andlet

$$P_n(p)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right){}^nC_k p^k(1-p)^{n-k}.$$

ThenP_n(p)->∞f(p)asn->∞andtheconvergenceisuniforminp.P_n(x)iscalledtheBernsteinPolynomials.

Theorem8.5

AWLLNfor independentbutnotnecessarilyidenticalrandom variables.

Let$X_1,X_2,\ldots X_n$beindependentrandom variableswithall

$E((X^2)<\infty$ and let $S_n=X_1+X_2+\ldots+X_n$ and $\mu=E\left(\frac{S_n}{n}\right)$, then

$$P\left\{\left|\frac{S_n}{n}-\mu_n\right|\ge\epsilon\right\}\le\frac{1}{\epsilon^2 n^2}\{Var(X_1)+Var(X_2)+\cdots+Var(X_n)\}.\varepsilon>0.$$

When $VarX_i\le M$ for all I. that is when the variances are all bounded bya constant which does not depend on n, then we have

$$P\left\{\left|\frac{S_n}{n}-\mu_n\right|\ge\epsilon\right\}\le\frac{M}{\epsilon^2 n}\text{—> } 0 \text{ as } n\text{—> }\infty.$$

3.4TheCentralLimitTheorem

The central limit theorem is a fundamental result in the theory and application of probability.Thelawoflargenumbersassertsthat $\frac{S_n}{n}$ inaseriesofnindependenttrailswithconstantprobabilitypofsuccessinonetrailtends(inacertainprobabilisticsense)topasnincreases.Butthisassertiondoesnottellusanythingaboutthe distributionof $\frac{S_n}{n}$

asnbecomeslarge.Theanswertothisquestionisgivenbythesocalledcentrallimittheorem.T hetheoremassertsthatunderquitegeneralconditionsthesumofindependent variableshastheNormaldistribution in thelimit.

Theorem8.6:Centrallimittheorem

Let$X_1, X_2, \ldots, X_n$ beasequenceofindependentrandomvariableswith$E(X_i)=\mu$and$Var(X_i)= \sigma_i^2$ I= 1,2,…Let$S_n= X_1+X_2+\ldots+X_n$.Thenunder generalconditions,

$$Z_n = \frac{S_n - E(S_n)}{\sqrt{Var\ S_n}} = \frac{S_n - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}$$

Has approximatelythestandardNormaldistribution.

TheIdenticallyDistributionCase

Theorem8.7

Let$X_1, X_2, \ldots, X_n$be independentidenticallydistributedrandomvariables.Then

(i) $Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ (ii) $Z_n = \frac{\frac{S_n}{n} - \mu}{\sqrt{\sigma^2/n}}$

tendto thestandardnormaldistributionas$n \to \infty$.

WhatisremarkableaboutthistheoremisthatallweneedtoknowaboutXisitsmean andvarianceand thereisno needtospecify thenatureofitsdistribution.Also$\underline{X}$canbe discrete,continuousorboth.The outlineof theproof isgiven below,

Proof:

Let $M_X(t) = E(e^{tX})$, $S_n(t) = [E(e^{tX})]^n$

Let $Z_n = \frac{sn - n\mu}{\sqrt{n\sigma^2}} = \frac{sn}{\sqrt{n}\sigma} - \sqrt{n}.\frac{\mu}{\sigma}$.

The m.g.f of Z_n is

$$M_{Z_n}(t) = e^{(\sqrt{n}\mu/\sigma)t}\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

Taking log we have

$$In\ M_{Z_n}(t) = -\sqrt{n}\frac{\mu}{\sigma}t - n\ In\ M_X\left(\frac{1}{\sigma\sqrt{n}}\right).$$

Using Maclaurin series

$$M(t) = 1 + M'(0)t + M''\frac{(0)t^2}{2!} + \cdots$$

From above we have

$$In\ M_{Z_n}(t) = -\sqrt{n}\frac{\mu t}{\sigma}t + n\ In\left[1 + \frac{\mu t}{\sigma\sqrt{n}} + \frac{(\mu^2 - \sigma^2)t^2}{2n\sigma^2} + A\right] \qquad (7)$$

where A is the remainder term from Maclaurin series. Similarly, when the Maclaurin series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

is used to expand $\ln\left(1 + \frac{\mu t}{\sqrt{n}\sigma} + \frac{(\mu^2 + \sigma^2)}{n\sigma^2}t + A\right)$. $setting\ x = \frac{\mu t}{\sqrt{n}\sigma} + \frac{(\mu^2 + \sigma^2)t^2}{2n\sigma^2} + A,$

we find that

$\ln\left(1 + \frac{\mu t}{\sqrt{n}\sigma} + \frac{(\mu^2 + \sigma^2)}{n\sigma^2}t + A\right)$

$= \frac{\mu t}{\sqrt{n}\sigma} + \frac{(\mu^2 + \sigma^2)}{n\sigma^2}t + A - \frac{1}{2}\left[\frac{\mu t}{\sqrt{n}\sigma} + \frac{(\mu^2 + \sigma^2)}{n\sigma^2}t + A\right]^2 + \cdots$

It is left as an exercise to show that for large n.

$$\left|\frac{\mu t}{\sqrt{n}\sigma} + \frac{(\mu^2 + \sigma^2)}{2n\sigma^2} + A\right| < 1.$$

Onsubstitutingin(7)weobtain

$$In\ M_{Z_n}(t) = -\sqrt{n}\frac{\mu t}{\sigma}$$
$$+ n\left\{\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\mu^2 - \mu^2)}{2n\sigma^2} + A - \frac{1}{2}\left[\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\mu^2 + \sigma^2)t^2}{2n\sigma^2} + A\right]^2 + \ldots\right\}$$

It can easily be shown that as n -> ∞

$$In\ M_{Z_n}(t) = \frac{t^2}{2}.$$

Hence,

$$\lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2},$$

the moment generating function of a random variable having the standard normal distribution.

Theorem

LetX_n-n>1,andXberandomvariablessuchthat$\lim_{n\to\infty} M_{X\infty}(t) = M_X(t), -\infty < t < \infty$.

Then Lim $FXnx = FXx$ *at all points where* FXx is continuous.

Example8.2

Aboythrowsafairdie100times.Whatistheprobabilitythathismeanscorewillexceed3.

LetXrepresenttheoutcomeof eachtoss. PossiblevaluesofXare1,2,3,4,5,6. E(X)= 3.5.Var(X)=2.92.

Thatis$\mu=3.5.\sigma^2=2.92$

LetXbethemeanscore.BythecentrallimittheoremXisnormallydistributedwith mean3.5andVarσ^2/n.

$$\frac{\sigma^2}{n} = \frac{2.92}{100}.$$

Hence,

$$Z = \frac{\bar{X} - 3.5}{\sqrt{0.029}}$$

has a standard normal distribution.

$$P(\bar{X} > 3) = P\left(Z > \frac{-0.5}{0.170}\right) = P(Z > -2.94) = 0.9984.$$

Example8.3

Marks in an I.Q examination are normally distributed with mean 55 and standard deviation10.Whatistheprobabilitythat

(i) themeanmarkof agroupof 10studentswillbeabove50
(ii) themeanmarkof agroupof 20studentswillbebetween40 and50.
(iii) thesumofthemarksof the10studentswillbelessthan500.

Solution

Let$X_1,X_2,\ldots X_{10}$be themarkscoredbyeachstudentrespectively.

$E(X_1)= E(X_2)=\ldots=E(X_{10})= 55$

$\sigma_{X_1} = \sigma_{X_2} = \sigma_{X_n} = 10.$

(i) $\bar{X}$ = mean, be the central limit theory (assuming n = 10 is large enough for this distribution) $\frac{\bar{X}-55}{\sigma/\sqrt{n}}$ has a standard Normal distribution.

$$\frac{\bar{X}-55}{10/\sqrt{10}} = \frac{\bar{X}-55}{\sqrt{10}}$$

$$P(\bar{X} > 50) = P\left(\frac{\bar{X}-55}{\sqrt{10}} > \frac{50-55}{\sqrt{10}}\right) = 0.9306.$$

(ii) The mean mark of a group of 20 students between will be between 40 and 50. In this case, n = 20.

$$P(40 < \bar{X} < 50) = P(\bar{X} < 50) - P(\bar{X} \le 40)$$

$$P(\bar{X} \le 40) = P\left(z \le \frac{40-55}{\sqrt{5}}\right) = P(Z \le -3\sqrt{5}) = 0.00$$

$$P(\bar{X} < 50) = P\left(z < \frac{50-55}{\sqrt{5}}\right) = P(Z < -\sqrt{5}) = 0.01$$

Hence

$$P(40 < \bar{X} < 50) = 0.01 - 0.00 = 0.01$$

(iii) $S_{10} = X_1 + X_2 + \ldots + X_{10}$

$\frac{S_{10} - n\mu}{\sqrt{n\sigma^2}}$ has a standard Normal

$$\frac{S_{10}-550}{\sqrt{1000}} = \frac{500-550}{10\sqrt{10}} = \frac{-50}{10\sqrt{10}} = -1.53$$

Hence,

$$P(S_n < 500) = 0.057.$$

3.5NormalApproximatelytotheBinomialDistribution

Let$X_1, X_2, \ldots, X_n$benindependentidenticallydistributedBernoullitrailswh ere

$$X_i = \begin{cases} 1 & \text{with probability} P \\ 0 & \text{withprobability} 1 - P \end{cases}$$

$E(X_i)$=pandVar(X_i)=p(1– p).Fromtheorem2,thedistribution

$$\frac{S_n - E(S_n)}{\sqrt{n\sigma^2}} \text{--> standard Normal.}$$

where $S_n = X_1 + X_2 + \ldots + X_n$

$$E(S_n) = np$$

$$Var(S_n) = np(1-p),\ \sigma^2 = p(1-p).$$

Note that S_n has binomial distribution with mean np and variance np(1-P).

$$P(S_n \leq y) = \sum_{k=0}^{y} {}^nC_k\, P^k\, (1-p)^{n\text{-}k}$$

For large n

$$P(S_n \leq y) = P\left\{Z_n \leq \frac{y - np}{\sqrt{np(1-p)}}\right\}$$

$$\varphi = \left(\frac{y - np}{\sqrt{np(1-p)}}\right)$$

where Z_n is standard Normal random variable. Thus, from above we have

$$F_{S_n}(y) = P(S_n \leq y) = \sum_{k=0}^{y} {}^nC_k p^k (1-p)^{n\text{-}k} = \varphi\left(\frac{y - np}{np(1-p)}\right)$$

Example8.3

Itis claimedthat60% of thevotersI agivenward aregoingtovotefor partyA.assuming thatallvoterswillvote,andthatthereare100voters.WhatistheprobabilitythatParty A receivesatleast50votes.
Letp=0.6.prob.Of votingfor A.n=100.Assuming independentvoting

$$P(S_n > 50) = \sum_{k=50}^{100} 100C_k p^k (1-p)^{100\text{-}k} = \sum_{k=50}^{100} 100C_k (0.6)^k (0.4)^{100\text{-}k}$$

Using normal approximation (6) above, we have

$$P(S_n > 50) = P\left(Z \geq \frac{50 - 100 \times 0.6}{\sqrt{100(0.6)(0.4)}}\right) = P\left(Z \geq \frac{50 - 60}{\sqrt{24}}\right) = 0.9793$$

3.6 TheNormalApproximationtothePoissonDistribution

IfarandomvariableXhasapoissondistributionwhosemeanisλ,thenforlargeλthe standardizedrandomvariable.

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

hasastandardNormalDistribution.youwillrecallthatE(X)=λ,Var(X)=λ,Continuitycorrectionmay beintroduced.

Ithas beenshownthattheapproximationisgoodforλ>5.

Example8.4

A system suffers random breakdown at a constant rate of 10 per month. Find the probabilitythattherewillbeatleast8breakdownsinanymonth.

LetXbethenumberoftimethesystembreaksdowninanymonth.ThenXhasapoissondistributionwithλ=10.Thus

$$P(X > 8) = \sum_{k=8}^{\infty} \frac{e^{10}(10)^k}{k!}$$

Using Normal approximation, we have

$$\frac{X-10}{\sqrt{10}} \text{ has } N(0.1)$$

$$P(X \geq 8) = P\left(Z > \frac{8-10}{\sqrt{10}}\right) = P\left(Z > \frac{-2}{\sqrt{10}}\right) = 0.7357$$

SELF-ASSESSMENT EXERCISE8

i. Itisclaimedthat30%ofthevotersinagivenlocalgovernmentareaaregoingto voteforparty Ainalocalgovernmentelection.Assumingthatallvoterswillvote, there are4000 votersand theclaimisbasedonaproper(unbiased)sampling method.WhatistheprobabilitythatPartyAwillreceivemorethan1,500 votes.

ii. (a)Suppose thatasystemconsistsofcomponentseachof whichhasaprobabilityof0.05offailingduringaspecifictime.Thesystemfunctionsproperly when at least 150 components function. Assuming thesecomponentsfunctionindependentlyofoneanother,whatistheprobabilitythatthe system functionsproperlyduringaspecifictime.

(b) Suppose that the above system is made of n components each havingprobabilityof0.90ofnotfailingduringagiventime.Thesystemwillfunctionifatleast80 percentofthecomponentsfunctionproperly. Determine n so that the probability that the system functions properly duringaspecifictimeis0.96.

6.0 TUTOR-MARKED ASSIGNMENT

1. Let X be a non-negative interger-valued random variable whose probability generatingfunction$P_X(s)=E(s^X)$isfiniteallls.UseChebychev'sinequalitytoverifythefollowinginequalities:

   (a) $P(X \leq k) \leq \frac{P_X(s)}{s^k}. 0 \leq s \leq 1, k$ any positive inequalities:

   (b) $P(X > k) \leq \frac{P_X(s)}{s^k}. s > 1$

2. ProvetheChebychev'sinequality.

   LetXbearandomvariable(discreteorcontinuous)withE(X)=μandVar(X)=σ^2.Thenfor anypositiveconstantnumberkwehave

$P\{|X - \mu| \geq k\,\sigma\} \leq \frac{1}{k^2}$.

3. Provethat(i)$1 – x \leq e^{-x}$forrealx,(ii)$\log x \leq x-1$ for $x>0$,and

(iii) $\prod_{i=1}^{N} X_i \leq e^{-\sum_{i=1}^{N} X_i}$ $if\ X_i \leq 1, N = 1,2,\ldots$

7.0 REFERENCE/FURTHER READING

Harry Frank & Steven C. Althoen (1995). Statistics: Concepts and Applications. Cambridge University Press.