MODULE 1 FUNCTIONS, CONTINUITY AND CONVERGENCE OF SEQUENCE AND SERIES

UNIT 1 FUNCTION OF COMPLEX VARIABLES

CONTENT

1.0 INTRODUCTION

The set of real number is not adequate to handle some of the numbers we come across in mathematics. We needanother set – the complex numbers.

In this course we will do analysis on complex variables and establish those results which are analogue to the real number systems.

2.0 OBJECTIVES

At the end of this unit, youshould be able to:

- explain variables and functions of complex number; and
- solve problems on functions and transformation or complex variables.

3.0 MAIN CONTENT

3.1 Revision of Elementary Vector Algebra

A symbol, such as z, which can stand for any complex number is called a <u>complex variable</u>. If to each value a complex variable z can assume there correspondence one or more values of a complex variable w, we say that W is a function of z and write $w = f(z)$ or $w = g(z)$ etc. The variable z is sometimes called an independent variable while w is called a dependent variable. The value of a function at $z = a$ is often written as $f(a)$.

e.g. $f(z) = z^z$, for $z = 3i$, $f(z) = f(3i) = -9$. If one value of w corresponds to each value of z, we say that w is a single-valued function of z or that $f(z)$ each value of z, we say w is a multiple-valued or many-valued function of z.

Example 1: if $w = z^3$, then to each value of z there is only one value of w. $w = f(z) = z^3$ is a single-valued function of z.

3.2 Transformations

If $w = u + iv$ (where u and v are real) is a single-valued function of $z = u = iy$ (where x and y are real), we can write $u + iv = f(x + iy)$. By equating real and imaginary parts this is equivalent to

$u = u(x, y),\ v = v(x, y)$...(1)

Hence, given a point (x, y) in the z –plane, there corresponds a point (u, v) in the w plane. The set of equations (1) [or the equivalent, $w = f(z)$] is called a transformation.

Example 1

If $w = z^2$, then

$f(z^2) = (x + iy)^2 = x^2 - y^2 + 2xy$

Hence, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$

Example 2

Let $w = f(z) = \frac{1}{z}$ for z (.)

$$f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^3}$$

Hence,

$u(x,y)=\dfrac{x}{x^2+y^2}$ and $v(x,y)=\dfrac{-y}{x^2+y^2}$

3.3 The Elementary Functions

1 **Polynomial Functions**: Polynomial functions $P(z)$ are defined as $P(z)=a_0z^\eta+a_1z^{\eta-1}+................+a_{\eta-1}z+a_\eta$ where $a_0\neq 0,\quad a_1,.........,a_n$ are complex constants and η is a positive integer called the degree of the polynomial $P(z)$.

2. **Rational Algebraic Function** are defined by $F(z)=\dfrac{P(z)}{\vartheta(z)}$ where $P(z)$ and $\vartheta(z)$ are polynomials.

3 **Exponential Functions** are defined by

$w=f(z)=\lambda^{x+iy}=\lambda^x$ (Cos y - i Sin y)
where e is the natural base of logarithms. (e=2.71828). complex exponential functions have properties similar to those of real exponential functions.

For example $\lambda^{z1}\bullet\lambda^{z2}=\lambda^{zi+z2},\lambda^{z1}/\lambda^{z2}=\lambda^{z1-z2}$

$\lambda^{z_1}\bullet\lambda^{z_2}=\lambda^{x_1}$ (Cos y_1+ i $\sin y_1$)$\bullet\lambda^{x_2}$ (Cos y_2+i Sin y_2)

$=\lambda^{x_1}+\lambda^{x_2}(\text{ Cos } y_1+i\,Sin\,y_1)\left(Cis\,y_2+i\,Siny_2\right)$

$=\ e^{x_1}-e^{x_2}\left(Cos\ yi+1\ Sin\,yi\right)\left(Cos\ y_2+1\ Sin\ y_2\right)$

$=e^{x_1}+n_2\left[Cos\ y_1\ Cos\ y_2+i\ Cos\ y_i\ \ Sin\ y_2+1Siny_1\ Cos\,y_2-Sin\ y_1\ Sin\,y_2\right]$

$=\ e^{x_1+x_2}\ \left[\left(Cos\ y_1\ Cos\ y_2-Sin\,y_1\ Sin\,y_2\right)+i\left(Cos\,y_1\ Sin\,y_2+Sin\,y_1\ Cos\,y_2\right)\right]$

$=\ e^{x_1+x_2}Cos\left(y_1+y_2\right)+i\ Sin\left(y_1+y_2\right)$

$=\ e^{z_1+z_2}$

Note that when $w=e^z$, the number w can be written as
$w=\rho e^{i\phi}$ where $\rho=e^x$ and $\phi=y$

If we think of $w=e^z$ as a transformation from z to the w plane, we thus find that any non zero point $w=\rho e^{i\phi}$ is the z- Log $\rho+i\phi$

Therefore the range of the exponential function $w=e^z$ id the entire nonzero point $w=pe^{i\phi}$ is actually the image of an infinite number of points in the z plane under the transformation $w=e^z$. For in general, ϕ may have any one of the values

$\phi = \Phi + 2n\dot{\underset{\wedge}{-}}(n = 0, \pm 1, \pm 2...)$ where Φ denotes the principal value of $\arg w$. It then follows that w is the image of all the points.

$z = \log\ \rho + i\Phi + i\pi \quad (1\ n = 0,\ \mu\ 1,\ \pm 2, ...)$

Example 3: find all values of z such that $e^z = -1$

Solution

$e^z = e^x e^{iy}$,and $-1 = 1e^{i\pi}$ so that

$e^x e^{iy} = 1e^{i\pi}$

By equality of two complex numbers in exponential form, this means that

$e^x - 1\ and\ y = \pi + 2n\pi$ where nis an integral.

$n = \log 1 = 0,$ then

$z = (2n+1)\pi \qquad (n = 0, \pm 1,\ \pm 2.....)$

Example 4: Find the values of z for which $e^{4z} = 1$

Solution:

$e^{4z} = i$

$e^{4x}.e^{4yi} = e\ 1.i$

So that, by equality, we have

$e^{yx} = e^0 \Rightarrow\ 4x = 0 \Rightarrow n = 0$ and

$4y = 2n\overline{\pi} + \frac{\pi}{2}$

$y = \frac{1^{n\pi}0}{2} + \frac{1\pi 0}{8}$ for $(\eta = 0,\ \pm 1,\ \pm 2, ...)$

The solution is then $\frac{1}{2}n\pi\, i\, c \pm \frac{1}{8}\pi c$

SELF- ASSESSMENT EXERCISES

1 Show that (i) $\left|e^{z}\right| = e^{x}$ (ii) $e^{z+2k\pi i}$

2 Find the value of z for which $e^{3z} = 1$

4 Trigonometric Functions: are defined in terms of exponential functions as follows:

$$Sin\, z = \frac{e^{iz} - e^{-1z}}{2i}, \qquad Cos\, z = \frac{e^{iz} + e^{-iz}}{2}$$

$$Sec\, z = \frac{1}{Cos\, z} = \frac{2}{e^{iz} + e^{-iz}}, \qquad CSC\, Z = \frac{1}{Sin\, z} = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{Sin\, z}{Cos\, z} = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}, \qquad Cot\, z = \frac{Cos\, z}{Sin\, z}$$

$$= \frac{i\left(e^{iz} + e^{-iz}\right)}{e^{iz} - e^{-iz}}$$

Many properties satisfied by real trigonometric functions are also satisfied by complex trigonometric function.

e.g.

$$Sin^2 z + Cos^2 z = 1, \quad 1 + 1\tan^2 z = Sec^2 z, \quad 1 + wt^2 z = \csc^2 z.$$

$$Sin(-z) = -Sin\, z \quad Cos(-z) = Cos\, z \quad \tan(-z) = -\tan z$$

$$Sin(z_1 \pm z_2) = Sin\, z_1\ Cos z_2 \pm Cos\, z_1\ Sin\, z_2$$

$$Cos(z_1 \pm z_2) = Cos\ z_1\ Cos\, z_2 \pm Sin\, z_1\ Sin\, z_2$$

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 - \tan z_1\ \tan z_2}.$$

Activity 1

Prove that $Sin^2 z_0 + Cos^2 z_0 = 1$

Proof

By definition, $Sin\, z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $Cos\ z = \dfrac{e^{iz} + e^{-iz}}{2}$

Then $Sin^2 z + Cos^2 z = \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2$

$$= \left(\frac{e^{2iz} - 2 + e^{-2iz}}{4}\right) + \left(\frac{e^{2iz} + 2 + e^{-2iz}}{4}\right)$$

$$= 1.$$

6. Hyperbolic Function: Are defined as follows:

$$Sin\,hz = \frac{e^z - e^{-z}}{2},\ \ Coshz = \frac{e^z + e^{-z}}{2},\ \ \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$
$$Sechz = \frac{1}{Coshz},\ \ Co\sec hz = \frac{1}{Sinhz},\ \ Coth = \frac{Coshz}{Sinhz}$$

The following properties hold:

$$Cosh^2 z - Sinh^2 z = 1,\ \ 1 - \tanh^2 z = \sec h^2 z,\ Coth^2 z - 1 = \csc h^2 z$$
$$Sin\,h(-z) = -Sinhz,\ \ Cosh(-z) = Cosh\,z,\ \tanh(-z) = -\tanh(z)$$
$$Sinh(z_1 \pm z_2) = Sinhz_1\ Coshz_2 \pm Cos\,hz_1\ \ Sin\,hz_2$$
$$Cosh(z_1 \pm z_2) = Cosh\,z_1\ Coshz_2 \pm Sinh\,z_1\ Sinh\,z_2$$
$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}.$$

These properties can easily be proved from the definitions. For example, to show that:

$Cos\,h^2 z - Sin\,h^2 z = 1$, we observed that,

$$Cosh^2 z - Sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$
$$= \tfrac{1}{4}\left(e^{2z} + 2e^2 e^{-z} + e^{-2z}\right) - \tfrac{1}{4}\left(e^{2z} - 2e^z e^{-z} + e^2\right)^2$$
$$= \tfrac{1}{4}\left(e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}\right)$$
$$= \tfrac{4}{4} = 1.$$

Exercise:
The proofs of others are left as exercise

Trigonometric and hyperbolic functions are related. For instance:

$$Sin\,iz = i\,Sin\,hz,\ \ Cosi\,z = Cosh\,z,\ \ \tan i\,z = i\tanh z.$$
$$Sinhiz = i\,Sin\,z,\ \ Cos\,hiz = Cos\,z,\ \ \tan h\,i\,z = i\tan z$$

SELF- ASSESSMENT EXERCISES

1 If $Cos\,z = 2$, Find
 (a) $Cos\,2\,z$
 (b) $Cos\,3\,z$

2 Find $U(x, y)$ and $V(x, y)$ such that

(a) $Sinh\, 2z = \mu + iv$

(b) $z\, Cosh\, z = \mu + iv$

3 Evaluate the following

(a) $Sinh(\frac{\pi}{8})i$

(b) $\cosh \frac{2n+1)}{2}\pi$

(c) $Tan \cosh \frac{\pi i}{2}$

4 Show that $\left| Tanh \frac{\pi(1+i)}{4} \right| = 1$

5 If $\tan z = u + iv$ show that

$$\mu = \frac{Sin\, 2u}{Cos\, 2u + Cos\, h\, 2\, y}, \quad v = \frac{Sin\, h\, 2\, y}{Cos\, 2u + Cos\, h\, 2\, y}$$

6 Logarithmic Functions

The Natural Logarithm Function is the reverse of the Exponential Function and can be defined as:

$w = 1n\, z = 1n\, r + 1(\phi + 2k\pi),\ k = 0, \pm 1, \pm 2.$

Where $z = re^{iQ} = re^{i(Q + 2k\bar{\pi})}$

$1 \neq z$ is a multiple valued function with the principled value. In $i + i\phi$ where $0 \leq \phi \leq 2\pi$ or its equivalent.

For $z = a^w$ where a is real, $w = \log,\ z\ \ where\ \ a > 0,$ and $a \neq 0,1$, in this case, $z = e^{w\, In\, a}$ and so $w = \frac{1nz}{In\, a}$.

Exercises

Evaluate

(1) In (-40

(2) In$\left(\sqrt{3 - i}\right)$

Solutions

(i) In (-4)

$z = -4 + 0i, \quad r = |z| = \sqrt{-4^7 + 0^2} = \pm 4.$

$\arg z = \tan^{-1}\frac{0}{4} = \tan^{-1} 0 = 0 = \pi = \pi + 2k$

In (-4) = In $\left[4e^{i(\pi+2k\pi)}\right] =$ In $4+\left(\bar{\pi} + 2k\bar{\pi}\right)i$ for $k = 0, \pm 1, \pm 2$.......

(ii) In $\left(\sqrt{3} - i\right)$

$z = \sqrt{3} + -u, \quad r = |z| = \sqrt{\sqrt{3^2} + (-1)^2} = 2.$

$\arg z = \tan^{-1}\frac{-1}{\sqrt{3}} = -\frac{26}{180}\pi = \frac{334\pi}{180} + 2\pi k \neq \frac{11\pi}{6} + 2k\pi$

In $\left(\sqrt{3} - i\right) =$ In $\left(2e\frac{11}{6} + 2k\pi\right) =$ In 2 $+\left(\frac{11\pi}{6} + 2k\pi\right)i$

SELF- ASSESSMENT EXERCISES

Evaluate

(1) In$\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

(2) In $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

(3) In $\left(\sqrt{3} - 2i\right)$

7 Inverse Trigonometric Functions

To define the inverse sine function $Sin^{-1}z$, we write $w = Sin^{-1}z$ when $z = Sin\,w$. That is

$w = Sin^{-1}z$, when $z = \frac{e^{iw} - e^{-iw}}{21}$

Which is equivalent to:

$\left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 = 0$.

This is quadratic in w^{iw}. Solving for e^{iw} one have

$e^{iw} = 1z + \left(1 - z^2\right)^{1/2}$

Taking logarithms of both sides and recalling that $w = Sin^{-1/z}$, we have

$Sin^{-1}z = -i\ In\left[iz - \sqrt{1 - z^2}\right]$

Which is a multiple-valued function with infinitely many values at each z.

Similarly,

$$Cos^{-1} z = -i \ \ In\left[z + i\sqrt{1-z^2}\ \right]$$

$$\tan^{-1} z = \ \frac{i}{2}\ In\left\{\frac{1+z}{1-z}\right\}$$

Which are also multiple valued functions.

Exercise

Find the values of $Sin^{-1} 2$

Solution

$$Sin^{-1} 2 = -i \ \ In\left[2i + \sqrt{1-2^2}\ \right)$$

$$= -i \ \ In\left(2i + \sqrt{3}i\right) = -2 \ \ In\ \left(2 + \sqrt{3}\right)i$$

$$= -i \ In(2 + j3)e^{\left(\bar{\wedge}/_2 + 2k\bar{\wedge}\right)i}$$

$$= \ -i \ In\left(2 + j3\right) + \left(\frac{\pi}{2} + 2k\pi\right) i$$

$$-i \ In \ \left(2 + j3\right) + \frac{\pi}{2} + 2k\pi$$

SELF- ASSESSMENT EXERCISES

1. Evaluate
   (a) $Cos^{-1} 2$ (b) $\tan^{-1} 2$

8 Inverse Hyperbolic Functions

If $z = Sinhw$ then $w = Sinh^{-1} z$ is called the inverse hyperbolic sine of z. Other inverse hyperbolic functions are similarly defined.

$$Sin\,h^{-1} z = In\ \left\{z + \sqrt{z^2}\ + 1\right\}$$

$$Cos\,h^{-1} z = In\left\{z + \sqrt{z^2}\ - 1\right\}$$

$$\tan h^{-1} z = \ \frac{1}{2}\ In\left(\frac{1+z}{1-z}\right)$$

In each case, the constant $2k\bar{\wedge}i$, k = 0, $\pm 1, \pm 2...$ has been omitted. They are all multiple valued functions.

$$Cos\,h^{-1}i = In\ \{i + \sqrt{-1-1}\ =\ In\{i = \sqrt{2i}\}$$

$$= \text{In}(1 + j2)i\ =\ In(1+\sqrt{2})\exp\left\{\frac{\pi}{2} + 2n\pi\right\}i$$

$$= \text{In}\ (1 = \sqrt{2}) + i\frac{\pi}{2} + 2n\pi$$

SELF- ASSESSMENT EXERCISES

1. Find all the values of
   (a) $Sinh^{-1}\ i$
   (b) $Sin\,h^{-1}\ [In\ (-1)]$

4.0 CONCLUSION

In this unit we considered in general, functions of complexvariables and considered various functions in these categories. Practice all exercises in this unit to gain mastery of the topic.

5.0 SUMMARY

What we have learnt in this unit can be summarised as follows:

(a) Definition of Complex Variables

(b) Some Elementary functions of Complex Variables

(c) Transformation of Complex variables.

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that $Cos^{-1}z = -i\ In\ \left[z + i\ \sqrt{1-z^2}\right]$

2. Show that: In $(2-1) = \frac{1}{2}$ In $\{(u-1)^2 + y^2\} + i\tan^{-1}\frac{y}{x-1}$

3. Evaluate the following
   (a) $Sinh(\frac{\pi}{8})i$

(b) $\cosh \frac{2n+1)}{2}\pi$

(3) $Tan \cosh \frac{\pi i}{2}$

4. Show that $\left| Tanh \frac{\pi(1+i)}{4} \right| = 1$

5. If $\tan z = u + iv$ show that

$$\mu = \frac{Sin\, 2u}{Cos\, 2u + Cos\, h\, 2\, y}, \quad v = \frac{Sin\, h\, 2\, y}{Cos\, 2u + Cos\, h\, 2\, y}$$

7.0 REFERENCE/FURTHER READING

Francis, B. Hildebrand.(1976).*Advanced Calculus for Application.*(2nd ed.).

UNIT 2 LIMITS AND CONTINUITY OF FUNCTION OF COMPLEX VARIABLES

CONTENTS

1.0 INTRODUCTION

In this unit, we will learn about limits and continuity in complex variables,

We shall establish some relevant theorems on limits and continuity.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explainlimit and continuity of functions of complex variables,
- state theorems related to limits and continuity of complex variables, and
- solve all related questions on limits and continuity.

3.0 MAIN CONTENT

3.1 Limits

Definition: Let a function f be defined at all point Z in some neighborhood Z_0, except possibly for the point Z_0 itself. A complex number L is said to be the limit of $f(z)$ as Z approaches Z_0 if for each positive number ε there is a positive number δ such that $|f(z)-L|<\varepsilon$ whenever $0<|Z-Z_0|<\delta$

We write

$$\lim_{z \to z} f(z) = L$$

Example: Show that

$$\lim_{z \to 2i} \left(2x + iy^2\right) = Atl \qquad z = x + iy$$

Solution

For each positive number ε. We must find a positive number ε such that $\left|2x + iy^2 - 4i\right| < \varepsilon$ whenever $0 < |z - 2i| < \delta$

To do this, we must write

$$\left|2x + iy^2 - 4i\right| \le 2|x| + \left|y^2 - 4\right| = 2|x| + |y-2|\ |y+2|$$

and thus note that the first of inequalities will be satisfied if
$2|x| < \frac{\varepsilon}{2}$ and $|y-2|\ |y+2| < \varepsilon/2$

The first of these inequalities is, of course, satisfied if $|x| < \varepsilon/2$. To establish conditions on y such that the second holds, we restrict y so that $|y-2| < \varepsilon$ and then observe that

$$|y+2| = |(y-2)+4| \le |y-2| + 4 < 5$$

Hence if $|y-2| < \min\ \{\varepsilon/10, 1\}$, if follows that $|y-2|\ |y+2| < \left(\frac{\varepsilon}{10}\right)5 = \varepsilon/2$

An appropriate value of δ is now easily seen from the conditions that $|x|$ be less than $\varepsilon/4$ and that $|y-2|$ be less then $\min\ \{\varepsilon/10, 1\}$
$\delta = \min\ \{\varepsilon/10, 1\}$

Note that the limit of a function $f(z)$ at a point z_0 if it exists is unique. Suppose that

$$\lim_{z \to z_0} f(z) = L_0 \text{ and } \lim_{z \to z_0} f(z) = L_1$$

Then for an arbitrary positive number δ_0 and δ_1 such that
$|f(z) - L_0| < \varepsilon$ whenever $0 < |z - L_0| \quad \delta_0$
and $|f(z) - L_1| < \varepsilon$ whenever $0 < |z - L_1| < \delta_1$

So if $0 < |z - z_0| < \delta$ where δ denotes the smaller of the two numbers δ_0 and δ_1, then
$|(f(z) - Lo) - (f(z) - L_1)| \le |f(z) - Lo| + |f(z) - L_1| < 2\varepsilon$
That is

$|L_1 - Lo| < 2\ \varepsilon$
But

$L_1 - L_0$ is a constant, and ε can be chosen arbitrarily small. Hence, $L_1 - L_0 = 0$ *or* $L_i = L_0$

Definition: The statement

$$\lim_{z \to \infty} f(z) = L_0$$

means that for each positive number ε there is a positive number δ such that
$|f(z) - L_0| < \varepsilon$ whenever $|z| < \dfrac{1}{\delta}$
That is, the point $L = f(z)$ lies in the ε *nbd* $|l - L_0| < \varepsilon$ of L_0 whenever the point z lies in the *nbd* $|z| > 1/\delta$ of the point at infinity.

Example: Observe that

$$\lim_{z \to \infty} \frac{1}{z^2} = 0$$

Since

$$\left|\frac{1}{z^2} - 0\right| < \varepsilon \text{ whenever } |z| < \frac{1}{\sqrt{\varepsilon}}$$

Hence $\delta = \sqrt{\varepsilon}$
When L_0 is the point at infinity and z_0 lies in the finite plane, we write

$$\lim_{z \to z_0} f(z) = \infty$$

If for each ε there is a corresponding δ such that $|f(z)| > \dfrac{1}{\varepsilon}$ whenever $0 < |z - z_0| < \delta$

Example: As expected

$$\lim_{z \to 0} \frac{1}{z^2} = \infty$$

for $\left|\frac{1}{z^2}\right| > \frac{1}{\varepsilon}$ whenever $0 < |z - \varepsilon| < \sqrt{\varepsilon}$

3.2 Theorems on Limits

Theorem 1: Suppose that

$f(z) = U|x, y| + V(x, y)$, $z_0 = x_0 + iy_0$ and $L_0 = u_0 + iv_0$

Then

$$\lim_{z \to z_0} f(z) = Lo \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If and only if

$$\lim_{(x,y) \to (x_0, y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0, y_0)} v(x, y) = v_0 \ldots\ldots(2)$$

Proof

Assume (1) is true, by the definition of limit, there is for each positive number ε, a positive number δ such that $|(u - u_0) + i(v - v_0)| < \varepsilon$ whenever $0 < |(n - n_0) + i(y - y_0)| < \delta$

Since $|u - u_0| \leq |(u - u_0) + i(v - v_0)|$ and

$(u - u_0) + i\,(v - v_0)$

$|v - v_0| \leq |(v - v_0) + i\,(v - v_0)|$,

It follows that

$|u - u_0| < \varepsilon$ and $|v - v_0| < \delta$

whenever

$0 < |(n + iy) - (n = iy_0)| < \delta$

which is statement (1), hence the proof

Theorem 2: Suppose that

$$\lim_{z \to z_0} f(z) = L_0 \text{ and } \lim_{z \to z_0} f(z) = L_0$$

then

$$\lim_{z \to z_0} [f(z) + f(z)] = l_0 + L_0$$

$$\lim_{z \to z_0} [f(z) f(z)] = l_0 L_0$$

if $L_0 \pm 0$,

$$\lim_{z \to z_0} \frac{f(z)}{f(z)} = \frac{l_0}{L_0}$$

Proof: (Left as exercise)

SELF- ASSESSMENT EXERCISES

1. Evaluate the following using theorems on limits:

   (a) $\lim_{z \to z+i} (z^2 + 10z - 15)$

   (b) $\lim_{z \to z2i} \frac{(4z+3)(z-1)}{z^2 - 2z + 4}$

   (c) $\lim_{z \to 2e^{\bar{\pi}c/3}} \frac{z^3 + 8}{z^4 + 4z^2 + 16}$

   (d) $\lim_{z \to zi} (iz^4 + z^2 - 10i)$

   (e) $\lim_{z \to e^{\bar{\pi}/4}} \frac{z^2}{z^4 + z + 1}$

Show that

$$\lim_{z \to 2e^{\bar{\pi}/3}} \frac{z^3 + \delta}{z^4 + 4z^2 + 16} = \frac{3}{8} - \frac{\sqrt{3}}{8} i$$

3.3 Continuity

Definition: A function f is continuous at a point z_0 if all the following conditions are satisfied.

(1) $\lim_{z \to z_0} f(z)$ exists

(2) $f(z_0)$ exists

(3) $\lim_{z \to z_0} f(z) = f(z_0)$

Note that statement (3) contains (1) and (2) and it say that for each positive number ε there exist a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

a function of complex variable is said to be contours in a region R if it is continuous at each point

Example: The function

$$f(z) = xy^2 + i(2x - y)$$

is everywhere in the complex plane because the component functions are polynomials on x and y and are therefore continuous at each point (n, y)

Example: If

$$f(z) = \begin{Bmatrix} z^2 & z \neq l \\ \\ u & z - 1 \end{Bmatrix}$$

$\lim_{z \to i} f(z) = -1$ But $f(i) = 0$. Hence, $\lim_{z \to i} f(z) \neq f(i)$

Therefore the function is not continuous at $z = 1$

Example: The function

$$f(z) = e^{xy} + i\ Sin\ (n^2 - 2ny^3)$$

is continuous for all z because of the continuity of the polynomials on n and y as well as the continuity of the exponential and sine functions.

Theorem on Continuity

1 if $f(z)$ and $g(z)$ are at $z - z_0$. So also are the functions $f(z) * g(z)$, $f(z) - g(z)$, $f(z)\, g(z)$, $f(z)/g(z)$, the last only $g(z_0) \pm 0$.

2 A function of a continuous function is $w = g[f(z)]$ is its $f(z)$ is its

3 If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$arealso in the region.

4 If a function $f(z)$ is in a closed region, it is bounded in the region, i.e. there exists a constant M such that $|f(z)| < M$ for all points z in the region.

SELF- ASSESSMENT EXERCISES

1 Let $f(z) = \frac{z^2+4}{z-2i}$ if $z \pm 2i$ while $f(2i) = 3+4i$

(a) Prove that $\lim_{z \to i} f(z)$ exists and determine its value

(b) Is $f(z)$ at $z = 2i$? Explain?

(c) Is $f(z)$ at point $z \neq 2i$?. Explain

2 Find all possible points of discontinuity of the following function

(a) $f(z) = \frac{2z-3}{z^2+2z+2}$

(b) $f(z) = \frac{3z^2+4}{z^2-16}$

(c) $f(z) = Cot\ z$

Answers

(a) $-1 \pm i$
(b) $\pm 2, \pm 2i$
(c) $k\pi, \ \ k \neq v, \pm, \pm 2.$

3 For what values of z are each of the following function continuous

(a) $f(z) = \frac{z}{z^2+1}$

(b) $f(z) = \frac{1}{Sin\, z}$

4.0 CONCLUSION

In this unit we have studied limits of functions, continuity of functions of complex variables in a manner similar to that of real variables. You are required to master them properly so that you can be able to apply them when necessary.

5.0 SUMMARY

Recall the following points;

- Continuity in Complex variables can be treated analogously as in the real variables
- If $f(z)$ is a continuous complex variable so also its real and imaginary parts.

- A complex function $f(z)$ is bounded if there exist a constant M>0 such that $|f(z)| < M$

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that

$$\lim_{z \to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - 1} = 4 + 4i$$

Is the function at $z - i$?

2. Factorised

(i) $z^3 + 8$

(ii) $z^4 + 4z^2 + 16$

(b) (i) Show that

$$\lim_{z \to 2e^{\bar{\pi}/3}} \frac{z^3 + 8}{z^4 + 4z^2 + 16} = \frac{3}{8} - \frac{\sqrt{3}}{8}\, i$$

(ii) Discuss the continuity of

$$f(z) = \frac{z^3 + 8}{z^4 + 4z^2 + 16} \text{ at } z = 2e^{\bar{\pi} i/3}$$

7.0 REFERENCE/FURTHER READING

Francis, B. Hildebrand (1976).*Advanced Calculus for Application.*(2nd ed.).

UNIT 3 CONVERGENCE OF SEQUENCE AND SERIES OF COMPLEX VARIABLES

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about sequences and series of complex variables. You will also learn about the convergence of these series and sequences.

All related theorems in real variables will be established for complex variables. We shall consider Taylor and Laurent series.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define convergence of sequences and series on complex variables; and
- solve related problems on series and sequence.

3.0 MAIN CONTENT

3.1 Definition

An infinite sequence of complex numbers, $z_1, z_2, ...z_n, ...$ has a limit z if for each positive number ε there exists a positive integral number such that
$|z_n - z| < \varepsilon$ whenever $n > n_0$.
If the limit exists, it is unique.

When the limit z exists, the sequence is said to converge to z; and we write

$$\lim_{n \to \infty} z_n = z$$

If the sequence has no limit, it diverges.

Theorem: Suppose that $Z_n = x_n + iy_n \ (n = 1, 2, ...)$ and $z = x + iy$. Then

$$\lim_{n \to \infty} z_n = z \quad \text{..(i)}$$

If and only if

$$\lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y \quad \text{......................................(ii)}$$

Proof:

Assume (i) is true, for each positive number ε there exists a positive integer number such that

$|(x_n - x) + i(y_n - y)| < \varepsilon$ Whenever $n > n_0$

But

$|x_n - x| \leq |(x_n - x) + i(y_n - y)|$

And

$|y_n - y| \leq |(x_n - x) + i(y_n - y)|$

Consequently,

$|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ whenever $n > n_{0i}$ and (3) are satisfied.

Conversely, form (3), for each positive number ε, there is positive numbers n_1 and n_2 such that

$|x_n - x| < \frac{\varepsilon}{2}$ whenever $n > n_1$

And

$|y_n - y| < \frac{\varepsilon}{2}$ whenever $n > n_2$

Hence if number is the larger of the two integers n_1 and n_2,

Then

$|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$ whenever $n > n_0$

But
$|(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|$,
And so
$|z_n - z| < \varepsilon$ whenever $n > n_0$

Which is condition (2)

Definition: An infinite series of complex numbers $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + ... + z_n + ...$
Converges to a sum S, called the sum of the series, if the sequence
$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + ... + z_n \; (N = 1,\, 2,\, ...)$ of partial sums converges to S, we then write $\sum_{n=1}^{\infty} z_n = S$

Note that since a sequence can have at most one limit, a series can have at must one sum, when a series does not converge, we say that it diverge,

Theorem: Suppose that $z_n = x_n + iy_n \quad (n = 1, 2, ...)$ and $S = X + iY.$ then
$\sum_{n=1}^{\infty} z_n = S$
If and only if
$\sum_{n=1}^{\infty} Xn = X \;\; and \;\; \sum_{n=1}^{\infty} Yn = Y$

Definition: An infinite sequence of single valued functions of complex variable
$U_1(z), U_2(z),\; U_3(z), ..., U_n(z), ...$
Denoted by $\{U_n(z)\}$, has a limit $U(z)$ as $n \to \infty$, if given any positive number ε we can find a number N (depending in general on both ε and Z) such that $|U_n(z) - U(z)| < \varepsilon$ for all $n > N$.

We write $\lim_{n \to \infty} U_n(z) = U(z)$. In such case, we say that the sequence converges or is convergent to $U(z)$.

If a sequence converges for all values of Z (points) in a region R, we call R the region of convergence of the sequence. A sequence which is not convergent at some value (point) Z is called divergent at Z.

Definition: The sum of $\{U_n(z)\}$, denoted by $\{S_n(z)\}$ is symbolised by

$U_1(z)+U_2(z)+ \quad \neq \sum_{n=1}^{\infty} U_n(z)$ is called an infinite series

If $\lim_{n \to \infty} Sn(z) = S(z)$, the series is said to be convergent and S(z) is its sum, otherwise the series is said to be divergent. If a series converges for all values of Z (points) in a region R, we call R the region of convergence of the series.

Definition (absolute convergence):A series $\sum_{n=1}^{\infty} U_n(z)$ is called absolutely convergent of the series of absolute values.

i.e. $\sum_{n=1}^{\infty} |U_n(z)|$, converges

If $\sum_{n=1}^{\infty} U_n(z)$ converges but $\sum_{n=1}^{\infty} |U_n(z)|$ does not converge, we say that $\sum_{n=1}^{\infty} U_n(z)$ is conditionally convergent.

Definition: In the definition, if a number N depends only on ε and not in Z, the sequence $U_N(z)$ is said to be uniformly convergent.

3.2 Taylor Series

Theorem (Taylor's Theorem): Let f be analytic everywhere inside a circle C with center at Z_0 and radius R. Then at each point Z inside C.

$$f(z) = f(z_0) + \frac{f^1(z_0)}{1}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + ... + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + ...$$

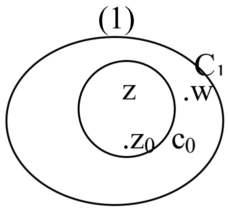
That is, the power series have converge to $f(z)$ when $|z - z_0| < R$.

Proof

Let Z_0 be any point inside C. Construct a circle C, with centre at z_0 and enclosing Z. Then by Cauchy's integral formula

$$f(z) = \frac{1}{\lambda ni} \oint_a \frac{f(w)}{w - z} dw$$

For any point w on C_1



We have

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{(w-z_0)}\left\{\frac{1}{1-\frac{(z-z_0)}{w-z_0}}\right\}$$

$$= \frac{1}{w-z_0}\left\{1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+.........+\left(\frac{(z-z_0)}{w-z_0}\right)^{n-1}+\left(\frac{z-z_0}{w-z_0}\right)^n\right\}$$

Or $\frac{1}{w-z} = \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)} + \frac{(z-z)^2}{(w-z)^3} + ... + \frac{(z-z_0)^{n-1}}{(w-z_0)^2} + \left(\frac{z-z_0}{w-z_0}\right)\frac{1}{w=z}$

Proof

We first prove the theorem when $z_0 = 0$ and then extends to any z_0.

Let z be any fixed point inside the circle C, centred now at the origin. Then let $|z| = r$ and note that $r < R$ where R is the radius of C. Let S denote points lying on a positively oriented circle C_1about the origin with radius R_1 where $r < R_1 < R$; then $|S| = R_1$. Since Z is interior to C_1, and f is analytic within and on the circle, the Cauchy integral formula gives

$$f(z) = \frac{1}{\lambda\pi i}\oint_{c_1}\frac{f(s)ds}{s-z} \qquad \text{……………………………………………………(2)}$$

Now, we can write

$\frac{1}{s-z} = \frac{1}{s}\left[\frac{1}{1-(z/s)}\right]$ and using the first that

$\frac{1}{1-c} = 1 + c + c^2 + ... + c^{n-1} + \frac{c^n}{1-c}(n = 1, 2, ...)$ where C is any complex number other than unity. Hence

$\frac{1}{s-z} = \frac{1}{s}\left[1 + z/s + (z/s)^2 + ... + (z/s)^{n-1} + \frac{(z/s)^n}{1-(z/s)}\right]$ and consequently

$$\frac{1}{s-z}=\frac{1}{S}+\frac{1Z}{S^2}+\frac{1}{S^3}Z^2+,........+\frac{1}{}\frac{Z^{N-1}}{S^N}+\frac{Z^N}{(s-z)S^N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Multiply this equation through by $f(s)\big/ 2\wedge\bar{-i}$ and integrate wrt S, we have

$$\frac{1}{2\pi\bar{i}}\int_c\frac{f(s)ds}{s-z}=\frac{1}{\lambda\pi\bar{i}}\int_{c_1}\frac{f(s)}{s}ds+\frac{z}{\lambda\pi\bar{i}}\int_{c_1}\frac{f(s)ds}{s^2}+\frac{z^2}{\lambda\pi i}\int_{c_1}\frac{f(s)}{s^3}ds+\frac{z^{n-1}}{\lambda\pi\bar{i}}\int_{c_1}\frac{f(s)}{s^n}+$$

$$\frac{z^n}{\lambda\pi\bar{i}}\int_{c_1}\frac{f(s)ds}{(s-z)S^N}$$

In view of expression (2) and applying this equation

$$\frac{1}{\lambda\pi i}\int_{C_1}\frac{f(s)ds}{S^{n+1}}=\frac{1}{\lambda\pi i}\int_{C_1}\frac{f(s)ds}{(S-i)^{N+1}}=\frac{f^{(n)}(0)}{n1}$$

We can write the result as

$$f(z)=f(0)+\frac{f'(0)}{1!}z+\frac{f'(0)}{\partial!}z^2+........+\frac{f^{(n-1)}(0)}{(n-1)!}Z^{N-1}+f(z)$$

Where

$$f(z)=\frac{z^n}{\lambda\pi\bar{i}}\int_{c_1}\frac{f(s)ds}{(s-z)s^N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Recalling that $|z|=r$ and $|\delta|=R_1$, where $r<R_1$, we note that $|s-z|\geq|s|-|z|=R_1-r$

It follows from (4) that when M_1 denotes the maximum of $|f(s)|$ on C_1,

$$|\rho_n(z)|\leq\frac{r^n}{\lambda\bar{\pi}}\left(\left(\frac{M_1}{(R_1-r)R_1^N}\lambda\bar{\pi}R_1\,\frac{M_1R_1}{R_1-r}\left(\frac{r}{R_1}\right)^N\right)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

But $\left(\frac{r}{R_1}\right)<1$, and therefore

$$\lim_{n\to\infty}\rho_N(z)=0$$

So that

$$f(z)=f(0)+\frac{f'(0)}{1!}z+\frac{f'(0)}{\lambda!}z^2+.......+\frac{f^{(n)}(0)}{n!}z^n+\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

In the open disk $|z|<R$.

This is a special case, of (1) and it is called the MACLURIN SERIES.

Suppose now that f is as in the statement of the theorem, since $f(z)$ is analytic when $|z-z_0|<R$, the composite function $f(z+z_0)$ is analytic when $|(z+z_0)-z_0|<R$. But the

last inequality is simply $|z| < R$; and if we write $g(z) = f(z + z_0)$, the analyticity of g inside the circle $|z| = R$ implies the existence of a Maclurin series representation.

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \quad (|z| < R)$$

That is

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)} z_0}{n!} z^n$$

Using z by $z - z_0$ in this equation, we arrive at the desired Taylor series representation for $f(z)$ about the point z_0.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad (|z - z_0| < R).$$

Example: If $f(z) = Sin\, z$, then $f^{(2n)}(0) = 0 \quad (n = 0, 1, 2.........)$ and $f^{(2n+1)}(0) = (-1)^n \quad (n = 0, 1, 2,..........)$ hence

$$Sin\ z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)}, \quad (|z| < \infty)$$

The condition $|z| < \infty$ follows from the fact that the function is entire.

Differentiating each side of the above equation with respect to and interchanging the symbols for differentiation and summation on the right-hand side, we have the expression

$$Cos\ \ z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2\pi)}$$

!

Because $Sin\, h\, z = -i \sin(iz)$, replacing z by iz in each side of (∞) and multiply through the result by –c, we have

$$Sin\, h\, z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Differentiating each side of this equation gives

$$Cos\, h\, z = \sum_{n=0}^{\infty} \frac{Z^{2n}}{(2n)!}$$

3.3 Laurent Series

Theorem: Let C_0 and C_1denote two positively oriented circles centred at a point Z_0, where C_0 is smaller than C_1. if a function f is analytic on C_0 and C, and throughout the annular domain between them, then at each point Z is the domain $f(z)$ is represented by the equation.

$$f(z) = \sum_{n=0}^{\infty} C_{in}(z - z_0)^n + \sum_{n=1}^{\infty}$$

Where

$$a_n = \frac{1}{\lambda \overline{\pi} i} \int_{C_1} \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots\ldots..)$$

And

$$b_n \frac{1}{\lambda \overline{\pi} i} \int_{C_0} \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots\ldots\ldots\ldots)$$

The series here is called a Laurent series

We let R_0 and R_1 denote the radius of C_0 and C_1 respectively. Thus R_0 and R_1 and if f is analytic at every point inside and on C_1 except at the point Z_0 itself, the radius R_0 may be taken arbitrarily small, expansion (1) then valid when

$0 < |z - z_0| < R_1$

If f is analytic at all points inside and on C_1, we need only write the integral in expansion (3) as $f(z)(z - z_0)^{n-1}$ to see that it is analytic inside and on C_0. For $n - 1 \geq 0$ when n is a positive integer. So all the coefficient bn are zero, and because

$$\frac{1}{\lambda \overline{\pi} i} \int_{C_1} \frac{f(z)dz}{(z - z_0)^{n+1}} = \frac{f^{(n)}(z)}{n!} \quad (n = 0, 1, 2, \ldots\ldots)$$

4.0 CONCLUSION

In this unit we have established condition for convergence of series in complex variables. You are required to study this unit properly to be able to understand subsequent units.

5.0 SUMMARY

The following DEFINITIONS is hereby recalled, to stress the importance of convergence of series in complex variables

1. An infinite sequence of complex numbers, $z_1, z_2, \ldots\ldots\ldots z_n, \ldots\ldots$ has a limit z if for each positive number ε there exists a positive integral number such that

   $|z_n - z| < \varepsilon$ whenever $n > n_0$.

   If the limit exists, it is unique.

When the limit z exists, the sequence is said to converge to z; and we write

$$\lim_{n \to \infty} z_n = z$$

If the sequence has no limit, it diverges.

2. We have also stated theorems that can help us in proofing convergence of series.

3. The Taylor and Laurent series have been applied in treating convergence of series.

6.0 TUTOR-MARKED ASSIGNMENT

1. Expand the following complex variable using Taylor series about z=$\frac{\pi}{2}$

   (a) *Tanz* (b) *Cosz*

2 State the Laurent series for the above.

7.0 REFERENCE/FURTHER READING

Francis, B. Hildebrand (1976).*Advanced Calculus for Application*(2nd ed.).

UNIT 4 SOME IMPORTANT THEOREMS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider some related theorems on complex variables.We shall consider theorems on test of convergence of complex variables and shall also learn about singularities and classifications or singularities.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the important theorems on convergences of sequences and series of complex variables;
- classify singularities on complex variables; and
- solve problems on complex variables.

3.0 MAIN CONTENT

Theorem 1: The limit of a sequence, if it exists, is unique.

Theorem 2: Let $\{a_n\}$be a real sequence with the property that

(i) $a_{nti} \geq a_n$ or $a_{nt1} \leq a_n$

(ii) $|a_n| < \mathrm{M}(a_{nt1})$

Then $\{a_n\}$converges.

That is, every bounded monotonic (increasing or decreasing) sequence has a limit.

Theorem 3: A necessary and sufficient conditions that $\{U_n\}$converges is that given $\varepsilon > 0,$we can find a number N such that $\left|U_n - U_q\right| < \varepsilon$ for all $\rho > N,\, q > N.$This is called Cauchy's convergence criterion.

3.1 Special Tests for Convergence

Theorem 1: (comparison tests)

(a) If $\sum |V_n|$converges and $|U_n| \le |V_n|$, then $\sum U_n$ converges absolutely

(b) If $\sum |V_n|$diverges and $|U_n| \ge |V_n|$, then $\sum |U_n|$diverges but $\sum U_n$ may or may not converge.

Theorem 2: (Ratio Test)

(a) If $\lim_{n \to \infty} \left|\frac{U_{nt1}}{U_n}\right| = L,$ then $\sum U_n$ converges (absolutely)

(b) If $L < 1$and diverges if $L > 1$. If L – 1, the test fails.

Theorem 3: (nth Root Test)

(a) If $\lim_{n \to \infty} \sqrt[n]{|U_n|} = L$, then $\sum U_n$ converges (absolutely)

(b) If L < 1 and diverges if L > 1. If L = 1, the test fails

Theorem 4: (Integral Test)

(a) If $f(x) \ge 0$ for $x \ge a,$ then $\sum f(x)$converges or diverges if $\lim_{m \to \infty} \int_a^M f(x)dx$ converge diverges.

Theorem 5: (Raabe's Test)

(a) If $\lim_{n \to \infty} n\left(1 - \left|\frac{U_{nt1}}{U_n}\right|\right) = L,$then $\sum U_n$ converges (absolutely)

(b) If L > 1 and diverges or converges conditionally if L < 1.

(c) If L = 1, the test fails.

Theorem 6: (Gauss' Test)

If $\left|\frac{U_{nt1}}{U_n}\right| = 1 - \frac{L}{n} + \frac{C_n}{n^2}$ where $|C_n| < M$ for all $n > N,$then $\sum U_n$ converges (absolutely) if L > 1 and diverges or converges conditionally if L ≤ 1.

3.2 Theorems on Power Series

Note that a series of the form

$a_0 + a_1(z - z_0) + a_2(z - z_0) + ... = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called a power series in $z - z_0$

Theorem 1: A power series converges uniformly and absolutely in any region which lies entirely inside its circle of convergence.

Theorem 2: (Abel's Theorem)
Let $\sum a_n z^n$ have radius of convergence $\Re$ and suppose that z_0 is a point on the circle of convergence such that $\sum a_n z_0^n$ converges.

Then $\lim_{z - z_0} \sum a_n z^n = \sum a_n z_0^n$ where $z \to z_0$ from within the circle of convergence.

Theorem 3: If $\sum a_n z^n$ converges to zero for all Z such that $|z| < R$ where $R > 0$, then $a_n = 0$. Equivalently. If $\sum a_n z^n = \sum b_n z^n$ for all Z such that $|z| < R,$ then $a_n = b_n$.

3.3 Laurent Series

If a function ***f*** fails to be analytic at a point z_0, we cannot apply Taylor's theorem at that point. If is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$.

Theorem (Laurent Theorem): Let C_0 and C_1 denote two positively oriented circles centred at a point z_0, where C_0 is smaller than C_1. If a function f is analytic at C_0 and C_1, and throughout the annular domain between them, then at each point z in that domain $f(z)$ is represented by the expansion.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad ....(1)$$

Where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)dz}{(z - z_0)^{nt1}} \; (n - 0,1,2,..........) \quad ....(2)$$

And

$$b_n = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)dz}{(z - z_0)^{-nt1}} \; (n = 1,2,..........) \quad ....(3)$$

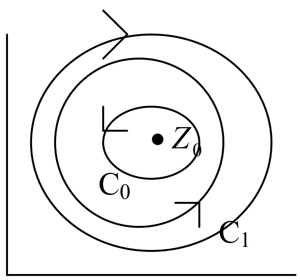
The series here is called a **Laurent series**.

Since the two integrands $\frac{f(z)}{(z-z_0)^{nt1}}$ and $\frac{f(z)}{(z-z_0)^{-nt1}}$ in expressions (2) and (3) are analytic throughout the annular domain $R_0|z-z_0|<R_1$, and in its boundary, any simple closed contour C around the domain in the positive direction can be used as a path of integration instead of the circular paths C_0 and C_1. Thus the Laurent series (1) can be written as

$f(z)=\sum_{n=-0}^{\infty} C_n(z-z_0)^n \left(R_0<|z-z_0|<R_1\right)$ Where

$C_n=\frac{1}{2\wedge i}\int_C \frac{f(z)dz}{(z-z_0)^{nt1}} \quad (n=0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots)$

Particular cases, of course, some of the coefficient may be zero.



Example: The expansion

$$\frac{e^z}{z^z}=\frac{1}{z^2}+\frac{1}{z}+\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+\ldots\ldots\ldots\ldots \quad 0<|z|<\infty$$

Follows from the Maclurin series representation

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots\ldots\ldots \quad (|z|<\infty)$$

3.4 Classification of Singularities

1 **Poles**: If $f(z)$ has the form

$f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots\ldots+\frac{a-1}{z-z_0}+\frac{a-2}{(z-z_0)^{2z}}$ In which the principal part has only a finite number of terms given by

$\frac{a_{+1}}{z-z_0}+\frac{a_{-2}}{(z-z_0)^2}+\ldots\ldots\ldots+\frac{a_{-n}}{(z-z_0)^n}$ Where $a_{-n}\neq 0$, then $z=z_0$ is called a pole of order n.

If n = 1, it is called a simple pole.

If $f(z)$ has a pole at $z=z_0$, then $\lim_{z\to z_0} f(z)=\infty$.

2 **Removable Singularities**: If a single valued function $f(z)$ is not defined at $z = z_0$ but $\lim_{z \to z_0} f(z)$ exist, then $z = z_0$ is a removable singularities. In such case, we define $f(z)$ at $z = z_0$ as equal to $\lim_{z \to z_0} f(z)$.

Example: If $f(z) = \frac{Sin\, z}{z}$, then $z = 0$ is a removable singularities since $f(0)$ is not defined but $\lim_{z \to 0} \frac{Sin\, z}{z} = 1$

Note that $\frac{Sin\, z}{z} = \frac{1}{z}\left\{ z - \frac{z^2}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} \right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} +$

3 **Essential Singularities**: If $f(z)$ is single valued, then any singularity which is not a pole or removable singularity is called an essential singularity. If $z = a\ u$ an essential singularity of $f(z)$, the principal part of the Laurent expansion has infinitely many terms

Example: Since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!}z^3$
$z = 0$ is an essential singularity.

4 **Branch Points**: A point $z = z_0$ is called a branch point of the multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged when Z describes a closed path about z_0. Since each of the branches of a multiple-valued function is analytic, all the theorems for analytic functions, in particular Taylor's theorem apply.

Example: The branch of $f(z) = z^{1/2}$ which has the value 1 for $z = 1$, has a Taylor series of the form

$a_0 + a_1(2-1) + a_2(z-1)^2 + ...$ With radius of convergence $R = 1$[the distance from Z=1 to the nearest singularity, namely the branch point z=0].

5 **Singularities at Infinity:** By letting $z = 1/w$ in $f(z)$ we obtain the function $f(1/w) = f(w)$. Then the nature of the singularity at $z = \infty$ [the point at infinity] is defined to be the same as that of $f(w)$ at $w = 0$.

Example: If $f(z)=z^3$ has a pole of order 3 at $z=\infty$, since $f(w)=f(1/w)=1/w^3$ has a pole of order 3 at $w=0$.
Similarly, $f(z)=e^z$ has an essential singularity at $z=\infty$, since $f(w)=f(1/w)=e^{1/w}$ has an essential singularity at $w=0$.

4.0 CONCLUSION

This unit is a very important unit which must be studied properly and understood before proceeding to other units.

5.0 SUMMARY

Recall that in this unit we discussed very important theorems in the solution of complex variables. We also discussed singularities, Laurent series and application, we discussed branch. These are to aid in tackling any exercises on complex variables.

6.0 TUTOR-MARKED ASSIGNMENT

1. State all the convergent tests listed in this unit

2. If $f(z)=\frac{Sinz}{z}$ determine the removable singularity and carry out the expansion.

3. Define the essential singularity and determine the essential singularity for $f(z)=e^{\frac{1}{z}}$

7.0 REFERENCE/FURTHER READING

Francis, B. Hildebrand (1976).*Advanced Calculus For Application*(2nd ed.).