MODULE 2 TAYLOR AND LAURENT SERIES, ANALYTIC FUNCTIONS AND COMPLEX INTEGRATION

UNIT 1 SOME EXAMPLES ON TAYLOR AND LAURENT SERIES

CONTENTS

1.0 INTRODUCTION

This unit considers examples on Taylor and Laurent series of complex variables.

The aim is to expose the students to more workable examples on complex variables.

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- solve problems successfully on complex variables using Taylor's Series and Laurent Series.

3.0 MAIN CONTENT

3.1 Examples on Taylor and Laurent Series

Example: Expand $f(z) = Cos\ z$ in Taylor series about $z = \frac{\pi}{4}$ and determine its region of convergence

Solution:

By Taylor series.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)\frac{(z - z_0)^2}{2!} + \ldots\ldots\ldots$$

$$f(z) = Cos\ z, \quad f'(z) = -Sin\, z, \quad f''(z) = -Cos\, z, \quad f'''(a) = Sin\, z$$

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}/2, f''\left(\frac{\pi}{4}\right) = -\sqrt{2}/2, \ f''\left(\frac{\pi}{4}\right) = -\sqrt{2}/2, \ f'''\left(\frac{\pi}{4}\right) = \sqrt{2}/2, \ldots\ldots.$$

$$f(z) = \sqrt{2}/2 - \sqrt{2}/2\left(z - \frac{\pi}{4}\right) - \frac{\sqrt{2}/2\left(z - \frac{\pi}{4}\right)^2}{2} + \sqrt{2}/2\frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \sqrt{2}/2\frac{\left(z - \frac{\pi}{4}\right)^4}{4!} - \ldots\ldots$$

$$f(z) = \sqrt{2}/2\left(1 - \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^2}{2} + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(z - \frac{\pi}{4}\right)^4}{4!} - \right)$$

$$f(z) = \sqrt{2}/2\left[\left(1 - \frac{\left(2 - \frac{\pi}{4}\right)^2}{2} + \frac{\left(z - \frac{\pi}{4}\right)}{4!} - \ldots\ldots\right) - \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(z - \frac{\pi}{4}\right)^5}{5!}\right]$$

$$\ldots\ldots\ldots\ldots(-1)^{n-1}\frac{\left(z - \frac{\pi}{4}\right)^{2n-2}}{(2_n - z)!}\ldots\ldots\frac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-1}}{(2n-1)!}$$

$$f(z) = \frac{\sqrt{2}}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-2}}{(2n-2)!} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-1}}{(2n-1)!}\right]$$

For the region of convergence, using ratio test

Let $U_n = \dfrac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-1}}{(2n-2)!}, \quad U_n + 1 = \dfrac{(-1)^2\left(z - \frac{\pi}{4}\right)^{2n}}{2n!}$

Also

Let $\dfrac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-1}}{(2n-1)!} = V_n, \ V_{n+1} = \dfrac{(-1)^n\left(z - \frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\left(z-\frac{\pi}{4}\right)}{2n} + \frac{(2n-2)!}{(-1)^{n-1}\left(z-\frac{\pi}{4}\right)^{2n-2}}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{(z-\frac{\pi}{4})^{2n}}{2n(2n-1)\left(z-\frac{\pi}{4}\right)^{2n-2}}\right|$$

$$= \lim_{n\to\infty} -\frac{1}{2n(2n-1)}\left|z-\frac{\pi}{4}\right|^2$$

$$= \lim_{n\to\infty} \frac{\left|z-\frac{\pi}{4}\right|^2}{-(2n(2n-1))} = 0$$

Similarly $\lim_{n\to\infty}\left|\frac{V_{n+1}}{V_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\left(z-\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} + \frac{(2n-1)!}{(-1)^{n-1}\left(z-\frac{\pi}{4}\right)^{2n-1}}\right|$

$$= \lim_{n\to\infty} -\frac{\left|z-\frac{\pi}{4}\right|^2}{2n(2n+1)} = 0$$

This shows that the singularity of $Cos\, z$ nearest to $\frac{\pi}{4}$ is at infinity. Hence, the series converges for all values of z i.e. $|z| < \infty$

Example: Expand $f(z) = \frac{1}{z-3}$ is a Laurent series valid for

(a) $|a| < 3$

(b) $|z| > 3$

Solution:

For $|z| < 3$

$$\frac{1}{(z-3)} = \frac{1}{-3+z} = \frac{1}{-3\left(1-\frac{z}{3}\right)} = \frac{1}{-3}\left(1-\frac{z}{3}\right)^{-1}$$

$$= -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}.......\right] = -\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\frac{z^3}{81}$$

For $|z| > 3$

$$\frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)} = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} = \frac{1}{z}\left[1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+.......\right]$$

$$= \frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\frac{27}{z^4}+........$$

Example: Expand $f(z) = \frac{z}{(z-1)(z-2)}$ in Laurent series valid for $|z|<|$

Solution:

$$\frac{z}{(z-1)(z-2)} = \frac{1}{z-1}+\frac{2}{2-z}$$

For $|z|<|$, $\frac{1}{z-1} = \frac{1}{1(1-z)} = -\left[1+z+z^2+z^3+z^4+....\right]$

$$= -1-z-z^2-z^3-z^4-$$

and

$$\frac{2}{2-z} = \frac{2}{2\left(1-\frac{z}{2}\right)} = 1\left(1-\frac{z}{2}\right)^{-1}$$

$$= 1+\frac{z}{2}+\frac{z^2}{4}+z\frac{3}{8}+\frac{z^4}{16}+.........$$

Adding, we have

$$\frac{z}{(z-1)(2-z)} = -\frac{1}{2}z-\frac{3}{4}z^2-\frac{7}{8}z^3-\frac{15}{16}z^4-...........$$

Example: Find the Laurent series for the function $f(z)=(z-3)\, Sin\frac{1}{z+2}$ about $z=-2$. Also state that type of singularity and the region of convergence for the series.

Solution:

$(z-3)\, Sin\frac{1}{z+2}$; $z=-2$. Let $z+2=u$ or $z=u-2$.

Then

$$(z-3)\, Sin\frac{1}{z+2}=(u-5)\, Sin\frac{1}{u}=(u-5)\left\{\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}+........\right\}$$

$$=1-\frac{5}{u}-\frac{1}{3!u^2}+\frac{5}{3!u^3}+\frac{1}{5!u^{4-}}$$

$$=1-\frac{5}{z+2}-\frac{1}{6(z+12)^2}+\frac{5}{6(z+12)^3}+\frac{1}{120(z+2)^4}.........$$

$z=-2$ is an essential singularity. The series converges for all values of $z+-2$.

4.0 CONCLUSION

In this unit, we discussed Laurent series and Taylor series. We applied them to solve some problems .You are to learn this unit very well.
You may wish to attempt the Tutor- Marked Assignment.

5.0 SUMMARY

Recall in this unit that while Taylor series can be useful to analyse functions, Laurent Series gives clearer and simple ways of handling functions of complex variables. These were clearly demonstrated in the examples considered in this unit. Answer the Tutor-Marked Assignment at the end of this unit, for more understanding of the concept.

6.0 TUTOR-MARKED ASSIGNMENT

1. Expand the function in each of the following series:
(a) a Taylor series of powers of z for $|z|<1$
(b) a Laurent series of powers of z for $|z|>1$
(c) a Taylor series of power of $z+1$ for $|z|<1$

7.0 REFERENCE/FURTHER READING

Hildebrand, Francis B. (2014). *Advanced Calculus for Application.* 6th Edition.

UNIT 2 ANALYTIC FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall study the analytic functions of complex variables. We shall establish the condition for functions to be analytic.

All related theorems on analytic function will be considered.

2.0 OBJECTIVES

At the end of this unit, you should have learnt about:

- derivatives of complex variables;
- Cauchy – Riemann equations;
- polar form of complex variables; and
- harmonic functions.

3.0 MAIN CONTENT

3.1 Derivatives

Definition: Let F be a …..whose domain of definition contains a nbd of a point Z_0. The derivative of f at Z_0, written as $f^1(Z_0)$, is defined as

$$f^1(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad ..(3.1.1)$$

Provided this limit exists. The function f is said to be differentiable at z_0 when its derivative at z_0 exists.

Note that (3.1.1) is equivalent to

$$f^1(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \ldots (3.1.2)$$

Where $\Delta z = z - z_0$

Which is also the same as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Where $f^1(z) = \frac{dw}{dz}$, $\Delta w = f(z_0 + \Delta z) - f(z_0)$ write $z - z_0$

Example: Suppose that

$$f(z) = z^2$$

At any point z

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Hence, $\frac{dw}{dz} = 2z$ or $f^1(z) = 2z$

Example: For the function $f(z) = |z|^2$

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \Delta \bar{z}) - z\bar{z}}{\Delta z}$$

$$= \bar{} + \Delta \bar{z} + z \frac{\Delta \bar{z}}{\Delta z}$$

When $z = 0$, this reduces to $\frac{\Delta w}{\Delta z} = \Delta \bar{z}$. Hence $\frac{dw}{dz} = \lim_{\Delta z \to 0} = 0$. at the origin $dw/dz = 0$

If the limit of $\Delta w/\Delta z$ exists when $z \neq 0$, this limit may be found by letting the variable $\Delta z = \Delta x + 1\Delta y$ approach 0 in any manner. In particular, when Δz approaches 0through the real values $\Delta z = \Delta z + i0$, we may write $\Delta \overline{Z} = \Delta Z$. Hence if the limit of $\Delta w/\Delta z$ exists, its value must be $\bar{z} + z$.

However, when ΔZ approaches 0 through the pure imaginary Value, so that $\Delta\overline{Z} = -\Delta Z$, the limit if found to be $\overline{Z} - Z$. Since a limit is unique, it follows that $\overline{Z} + Z = \overline{Z} - Z$, or $Z = 0$, if dw/dz exists. But $Z \neq 0$, and we may conclude from this contradiction that dw/dz exists only at the origin.

From example above, it follows that:

(1) A function can be differentiable at a certain point but nowhere else in any nbd of that point.

(2) Since the real and imaging parts of $f(z) = |z|^2$ are $\mu(n, y) = n^2 + y^2$ and $v(n, y) = 0$.
Respectively, it also shows that the real and imaginary components of a function of a complex variable can have continuous partial derivatives of all orders at a point and yet the function many not even be differentiable there.

(3) The function $f(z) = |z|^2$ is its at each point in the plane since its components functions are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there.

It is, however, true that the existence of the derivative of a function at a point implies the continuity of the function at that point.

3.2 Differentiation Formulae

Definition: Let F be a …..whose domain of definition contains a nbd of a point Z_0. The derivative of f at Z_0, written as $f^1(Z_0)$, is defined as

$$f^1(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{..(3.1.1)}$$

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Where $\Delta z = z - z_0$

Which is also the same as

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Where $f^1(z) = \frac{dw}{dz}$, $\Delta w = f(z_0 + \Delta z) - f(z_0)$ write $z - z_0$

Example: Suppose that

$$f(z) = z^2$$

At any point z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Hence, $\frac{dw}{dz} = 2z$ or $f^1(z) = 2z$

Example: For the function $f(z) = |z|^2$

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta 2)(\overline{z} + \Delta \overline{z}) - Z\overline{z}}{\Delta Z}$$

$$= \overline{Z} + \Delta \overline{Z} + Z \frac{\Delta \overline{Z}}{\Delta Z}$$

When z=0, this reduces to $\frac{\Delta w}{\Delta z} = \Delta \overline{Z}$. Hence $\frac{dw}{dz} = \lim_{\Delta z \to 0}$ = 0. at the origin $dw/_{dz} = 0$

If the limit of $\Delta w/_{\Delta z}$ exists when $z \neq 0$, this limit may be found by letting the variable $\Delta z = \Delta x + 1\Delta y$ approach 0 in any manner. In particular, when ΔZ approaches 0 through the real values $\Delta Z = \Delta n + i0$, we may write $\Delta \overline{Z} = \Delta Z$. Hence if the limit of $\Delta w/_{\Delta z}$ exists, its value must be $\overline{Z} + Z$.

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From example above, it follows that:

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3.3 Cauchy-Riemann Equations

Suppose that
$f(z)=u(x.y)+iv(x,y)$ and that $f^1(z_0)$ exists at a point $z_0=x_0+iy_0$. Then the first order partial derivatives of u and v wrt n and y must exist at (u_0, y_0), and they must satisfy.

$U_x(x_0y_0)=Vy(x_0,y_0)$ and $Uy(x_0,y_0)=-V_x(x_0,y_0)$ at that point. … (1)
Also $f^i(z_0)$ is given in terms of the partial derivatives by either
$f^1(z_0)=U_x(x_0,y_0)+iv(x_0y_0)$
or $f^1(z_0)=Vy(x_0,y_0)-Uy(x_0,y_0)$

Equation (1)… is referred to as Cauchy Riemann equation.

Example: the derivative of the function $f(z)=z^2$ exists everywhere.

To verify that the Cauchy-Riemann equations are satisfied everywhere, we note that
$f(z)=z^2=x^2-y^2+i2xy$ so that
$U(x,y)=x^2-y^2$ and $V_x(xiy)=2x$
$U_x(x,y)=2x$, $V_x(xiy)=2y$
$Uy(x,y)=2y$ $Vy(x,y)=2x$
So that
$U_n(n,y)=Vy(n,y)=2x$
$Uy(x,y)=-V_n(x,y)=-2y$
Also
$f^1(z)=U(x_0,y_0)+iV(x,y)=2x+i2y=2z$

3.4 Sufficient Conditions

Satisfaction of the Cauchy – Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. The following theorem gives sufficient conditions.

Theorem: (Sufficiency Theorem):
Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε- nbd of a point $z_0 = x_0 - iy_0$ suppose that the first-order partial derivatives of the functions U and V with respect to n and y exist everywhere in that nbd they are continuous at (x_0, y_0). Then, if these partial derivatives satisfy the Cauchy-Riemann equations.

$U_x = V_y, and, U_y = -Vx$
At (x_0, y_0), the derivative $f^1(z_0)$ exists.

Proof: We shall leave the proof as exercise.

Example: suppose that
$f(z) = e^x$ 9Cos y + i Sin y)

Where y is to be taken in radius when Cos y and Sin y are evaluated then

$U(x, y) = e^x Cosy$ and $V(x, y) = e^x Siny$

Since $U_x = V_y$ and $Uy = -Vx$ everywhere and since those derivatives are everywhere continuous, the conditions in the theorem are satisfied at all points in the complex plane. Thus, $f^1(z)$ exists everywhere and

$$f'(z) = U_x(x, y) = iV_x(x, y) = e^x(\cos x + i\sin y)$$

Note that $f^1(z) = f(z)$

Example: for the function
$f(z) = |z|^2 = U(x, y) = x^2 + y^2$ and $V(x, y) = 0$ So that $U_x(x, y) = 2x$ and $V_y(x, y) = 0$ while $U_y(x, y) = xy$ and $V_x(x, y) = 0$. Since $U_x(x, y) \neq V_y(x, y)$ unless $x = y = 0$ Cauchy-Riemann equations are not satisfied unless $x = y = 0$ the derivative $f'(z)$ cannot exist if $z \neq 0$ and besides, the existence of $f'(0)$ is not guaranteed unless conditions of theorem (3-4-1) are satisfied.

If follows from the theorem (3.4.1) that the further $f(z)=|z|^2=(x^2+y^2)+10$ has derivative at $z=0$; in fact, $f'(0)=0+0=0$.

3.5 Polar Form

Cauchy-Riemann equations can be written in polar form. For
$z=n+iy \ or \ z=r$ (Cos $\theta+i\,Sin\theta$), we have
$n=r\cos\theta, \quad y=r\sin\theta, \ r=\sqrt{n^2}+y^2. \ \theta=\tan^{-1} y/n$
Then,
$$Ur=Ur\frac{\partial r}{\partial r}+U\theta\frac{\partial\theta}{\partial x}=Ur\left(\frac{x}{\sqrt{x^2+y^2}}\right)+U\theta\left(\frac{-y}{x^2+y^2}\right)$$
So that
$$Uy=Ur\ Sin\theta+\frac{1}{r}\ U\theta\ Cos\ \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$
$$Vx=Vr\frac{\partial r}{\partial x}+V\theta\frac{r\theta}{rx}=Vr\ Cos\theta-\frac{1}{r}V\theta\ Sin\theta$$
So that
$$Vn=Vr\ Cos\theta=\frac{1}{r}V\theta\ Sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$
$$y=Vr\frac{\partial r}{\partial y}+V\theta\frac{\partial\theta}{\partial y}=Vr\ Sin\theta\frac{1}{r}\ V\theta\ \ Cos\ \theta$$
So that
$$Vy=Vr\ \ Sin\theta+\frac{1}{r}\ V\theta\ Cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From the Cauchy-Riemann equation, $Un=Vy$, equating (1) and (4), we have
$$\left(Ur-\frac{1}{r}V_\theta\right)\ Cos\theta-\left(Vr+\frac{1}{r}U_\theta\right)Sin\theta=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$
From the Cauchy-Riemann equation, $Uy=-Vn$, equating (2) and (3), we have
$$\left(Ur-\frac{1}{r}V_\theta\right)Sin\theta+\left(Vr+\frac{1}{r}U_\theta\right)Cos\ =0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Multiplying (5) by cos θ, (6) by Sin θ and adding given
$$(Ur=-\frac{1}{r}U_\theta) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Also, multiplying (5) by $-\sin\theta$, (6) by $Cos\theta$ and adding given
$$Vr=-\frac{1}{r}U\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Equations (7) and (8) are the Cauchy-Riemann equations in polar form.

Theorem: Let the function
$f(z) = U(r,\theta) + i\ v(r,\theta)$

Be defined throughout some ε neighborhood of a no zero point
$f(z) = r_0(Cos\ \theta_0 + i\,Sin\,\theta_0)$.

Suppose that the first order partial derivatives of the functions U and V wrt r and θ exist everywhere on that neighborhood and that they are continuous at (r_0, θ_0). Then if those partial derivatives satisfy polar forms (7) and (8) of the Cauchy-Riemann equations at (r_0, θ_0), the derivatives $f^1(z_0)$ exists.

The derivative $f^1(z_0)$ is given as

$$f^1(z_0) = e^{-1\theta}\left[Ur(r_0, \theta_0) + i\,Vr(z_0, \theta_0)\right]$$

Example: Consider the function
$f(z) = \frac{1}{r} = \frac{1}{re^{\theta}}$,
$U(r,\theta) = \frac{Cos\theta}{r}$ and $V(r,\theta) = \frac{-Sin\theta}{r}$ and the condition of the theorem are satisfied at any nonzero point $z = re^{i\theta}$ in the plane. Hence the derivative of f exists there: and according to (9)

$$f^1(z) = e^{-1\theta}\left(-\frac{Cos\theta}{r^2} + i\frac{Sin\theta}{r^2}\right) = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}$$

3.6 Analytic Functions

Definition: A function f of the complex variables z is analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighborhood of z_0. A function f is said to be analytic in a region R if it is analytic at each point in $\Re$. The term halomorphic is also used in literature to denote analyticity.

If $f(z) = z^2$, then f is analytic everywhere. But the function $f(z) = |z|$ is not analytic at any point since its derivative exists why at $z = 0$ and not throughout any nbd.

An entire function is a function that is analytic at each point in the entire plane. E.g. polynomial functions.

If a function f fails to be analytic at a point z_0, buy is analytic at some point in every nbd of z_0, then z_0 is called a singular point or singularity of f. For example, the function $f(z) = 1/z$, where derivative is $f(z) = -1/z^2$ is analytic at every point except $z = 0$ hence it is not even defined. Therefore the point $z = 0$ is a singular point.

If two functions are analytic in domain D, their sum and their product are both analytic in D. similarly, their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.

3.7 Harmonic Functions

A real-valued function h of two real variables x and y is said to be harmonic in a given domain in the xy plane if throughout that domain it has continuous partial derivatives of first and second order and satisfies the partial differential equation.

$h_{xx}(x, y) + h_{yy}(x, y) = 0$ ……………………………………………(3.7.1)

Known as LAPLACE'S EQUATION

If a function
$f(z) = u(x, y) + i\,v(x, y)$…………………………………………...(3.7.2)

Is analytic in a domain D, then its component functions U and V are harmonic in D. to show this,

Since f is analytic in D, the first order partial derivatives of its component functions satisfy the Cauchy-Riemann equations throughout D.
$Ux = Vy,\ Uy = -Vx$ ……………………………………………….(3.7.3)

Differentiating both sides of these equating with respect to x, we have
$Uxy = Vyy \quad Uyy = -Vxy$ …………………………………………… (3.7.4)

The continuity of the partial derivatives ensures that $Uyx = Uxy$ and $Vyx = Vxy$. It then follows from (3.7.4) and (3.7.5) that $Uxx(n, y) + Uyy(x, y) = 0$ and $Vxx(x, y) + Vyy(x, y) = 0$.

Thus, if a function $f(z) = U(x, y) + i\,V(x, y)$ is analytic in a domain D, its component functions U and V are harmonic in D.

3.8 Solved Problems

Example 1: Verify that the real and imaginary parts of the function $f(z) = z^2 + 5iz + 3 = i$ satisfy Cauchy-Riemann equation and deduce the analyticity of the function.

Solution:

$$f(z) = z^2 + 5iz + 3 - 1$$
$$= (x + iy)^2 + 5i(x + iy) + 3 = 1$$
$$= x^2 - y^2 - 5y + 3 + i(2xy + 5x - 1)$$

So that

$U(x, y) = x^2 - y^2 - 5y + 3, \qquad V(x, y) = 2xy + 5x - 1$

$Ux(x, y) = 2x, \ Uy(x, y) = -2y - 5 = -(2y + 5)$

$Vx(x, y) = 2y + 5 \qquad Vy(x, y) = 2x$

And since $Ux(x, y) = Vy(x, y) = 2x$

And $Uy(x, y) = -Vx = -(2y + 5)$

The function satisfies Cauchy Riemann equation. Also, since the partial derivatives are polynomial functions which are continuous, then the function is analytic.

Example 2: (a) Prove that the function $U = 2x(1 - y)$ is harmonic

(b) Find a function V such that $f(z) = u + iv$ and express $f(z)$ in terms of z.

Solution:

(a) $U = 2x(1 - y)$.

The function is harmonic if $Uxx + Uyy = 0$

$Ux = 2(1 - y), \quad Uxx = 0$

$Uy = -2x \qquad Uyy = 0$

$Uxx + Uyy = 0 + 0 = 0$. Hence the function is harmonic

(b) By Cauchy-Riemann equation

Example 3: show that the function $U(x, y) = y^3 - 3x^2 y$ is harmonic and find its harmonic conjugate.

Solution:

$U(x, y) = y^3 - 3x^2 y$
$Ux = -6xy, \quad Uxx = -6y$
$Uy = 3y^2 - 3x^2 \qquad Uyy = 6y$
And since
$Uxx + Uyy = -6y + 6y = 0$

The function
$U(x, y) = y^3 3x^2 y$ is harmonic

To find the harmonic conjugate,
From
$Ux(x, y) = -6xy$, since $Ux = Vy$,
$Vy(x, y) = -6xy$
Find x, and integrate both sides with respect to y,
$V(x, y) = -3xy^2 + \phi(x)$
And since $Uy = -Vx$ must hold, it follows from $(x)$and (xr)that
$3y^2 - 3x^2 = 3y^2 + \phi^1(x)$
So that
$\phi^1(x) = 3x^2$and $\phi(x) = 6x + C$
$V(x, y) = -3xy^2 + 6x + C$.
Is the harmonic conjugate of $x / x, y)$

The corresponding analytic function u
$f(z) = (y^3 - 3x^2 y) + i(x^3 - 3xy^2 + C)$
Which is equivalent to
$f(z) = i(z^3 + 1)$

SELF -ASSESSMENT EXERCISES

1. Verify that the real and imaginary parts of the following functions satisfy the Cauchy-Riemann equations and thus deduce the analyticity of each function
   (a) $f(z) = z^2 + 5iz + 3 = 1$
   (b) $f(z) = ze^{-z}$
   (c) $f(z) = Sin 2z$

2. (a) Prove that the function $U = 2x(1 - y)$ is harmonic
   (b) Find a function $v \;\; s.t \; f(z) = u + iv$ is analytic
   (c) Express $f(z)$ in terms of z

3. Verify that C – R equation are satisfied for the functions
   (a) e^{z^2}
   (b) $Cos\,2z$
   (c) $Sinh\,4z$

4. Determine which of the following functions are harmonic and find their conjugates.
   (a) $3x^2y + 2x^2 - y^3 - 2y^2$
   (b) $2xy + 3xy^2 - 2y^3$
   (c) $xe^x \cos y - ye^x Sin\,y$
   (d) $e^{-2xy} Sin\,(x^2 - y^2)$

5. (a) Prove that $\psi = In\left[(x - 1j^2) + (y - 2j^2)\right]$ is harmonic in every region which does not include the point (1, 2)
   (b) Find a function ϕ $s.\ t$ $\psi + i\psi.1$ analytic
   (c) Express $\psi\,x\,i\psi$ as a function of Z

6. If U and V are harmonic in a region R, prove that $(Uy - Vx) + i\ (Ux + Vy)$ is analytic in R.

4.0 CONCLUSION

This unit had been devoted to treatment of special class of function usually dealt with both in real and complex functions. You are required to master these functions so that you can be able to solve problems associated with them.

5.0 SUMMARY

Recall that in this unit we considered derivatives in complex variables, we derived the Cauchy Riemann equations for determining analytic functions in complex variables, we also studied harmonic functions etc. Examples were given to illustrate each of these functions.

6.0 TUTOR-MARKED ASSIGNMENT

1. (a) Prove that the function $U = 2x(1 - y)$ is harmonic
   (b) Find a function v $s.\,t$ $f(z) = u + iv$ is analytic
   (c) Express $f(z)$ in terms of z

2. Verify that C – R equation are satisfied for the functions
   (a) e^{z^2}
   (b) $Cos\, 2z$
   (c) $Sinh\, 4z$

3. Determine which of the following functions are harmonic and find their conjugates.
   (a) $3x^2 y + 2x^2 - y^3 - 2y^2$
   (b) $2xy + 3xy^2 - 2y^3$
   (c) $xe^x \cos y - ye^x Sin\, y$
   (d) $e^{-2xy} Sin\,(x^2 - y^2)$

4. (a) Prove that $\psi = In[(x-1j^2) + (y-2j^2)]$ is harmonic in every region which does not include the point (1, 2)
   (b) Find a function $\phi \;\; s.\, t \;\; \psi + i\psi.1$ analytic
   (c) Express $\psi\, x\, i\psi$ as a function of Z

5. If U and V are harmonic in a region R, prove that $(Uy - Vx) + i\;(Ux + Vy)$ is analytic in R

7.0 REFERENCE/FURTHER READING

Hildebrand, Francis B. (2014). Advanced Calculus for Applications. 6th Edition.

UNIT 3 PRINCIPLES OF ANALYTIC CONTINUATION

CONTENTS

1.0 INTRODUCTION

We shall examine in this unit principle of analytic continuation and establish conditions under which functions of complex variables will be analytic in some regions.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define residues and residues theorem;
- do calculations of residues; and
- answer questions on residues.

3.0 MAIN CONTENT

Suppose that inside some circle of convergence C_1 with centre at a, $f(z)$ is represented by a Taylor series expansion defined by:

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ...(1)$$

If the value of $f(z)$ is not known, choosing a point b inside C_1, we can find the value of $f(z)$ and its derivatives at b. from (1) and thus arrive at a new series

$$b_0 + b_1(z-b) + b_2(z-b)^2 + ...+ .... \qquad (2)$$

Having circle of convergence C_2. If C_2 extends beyond C_1, then the values of and its durations can be obtained in this extended portion.

In this case, we say that $f(z)$ has been extended analytically beyond C_1 and the process is called analytic continuation or analytic extension. This process can be repeated indefinitely.

Definition: Let $F_1(z)$ be a function of z which is analytic in a region R_1. Suppose that we can find a function $F_2(z)$ which is analytic in a region R_2 and which is such that $F_1(z) = F_2(z)$ in the region common to R_1 and R_2. Then we say that $F_2(z)$ is an analytic continuation of $F_1(z)$.

3.1 Residues and Residues Theorems

Recall that a point z_0 is called a singular point of the function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. A singular point z_0 is said to be isolated if in addition, there is some *nbd* of z_0 throughout which f is analytic except at the point itself.
When z_0 is an isolated singular point of a function f, there is a positive number R, such that f is analytic at each point z for which $0 < |z - z_0| < R_1$ consequently the function is represented by a series.

$$f(z) = \sum_{n=0} an(z - z_0)^\eta + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_\eta}{(z - z)^\eta} + ...$$

$$0 < |z - z_0| < R_1$$

Where the coefficients $a\eta$ and $b\eta$ have certain integral representations. In particular

$$b_\eta = \frac{1}{\eta\overline{\eta}i} \int_c \frac{f(z)dz}{(z - z_0)^{\eta+1}} \qquad (\eta = 1, \quad 2..........) \qquad ....(2)$$

When C is any positively oriented simple closed contour around Z_0 and lying in the domain $0 < |z - z_\eta| < R$
When $\eta = 1$, this expression for b_η can be written

$$\int_c f(z)dz = 2\eta i\, b_1 \qquad .... \qquad (3)$$

The complex number b_1 which is the coefficient of $\frac{1}{(z - z_0)}$ in expansion (1) called the residue of f at the isolated singular point z_0
Equation (3) provides a powerful method for conducting certain integral around simple closed ……

Example

Consider the integral

$$\int_c \frac{e^{-z}}{(z-1)^2}dz$$

Is analytic within and on C except at the isolated singular point z =1. Thus, according to equation (3), the value of integral (4) is ……times the …..of f at z = 1. To determine this residue, we recall the maclaurin series expansion.

$$e^z = \sum_{n=o}^{\infty}\frac{z^\eta}{n1} \qquad (|z| < \infty)$$

From which it follows that

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}e^{-(z-1)}}{(z-1)^2} = \sum_{n-0}^{\infty}\frac{(-1)^\eta(z-1)^{n-2}}{\eta\ i\ e} \quad (0 < |z-1| < \infty)$$

In this Laurent series expansion, which can be written in the form (1), the coefficient of $\frac{1}{z-1}$ is $^{-1}\!/_e$ that is, the residue of f at z = 1 is $^{-1}\!/_e$. Hence $\int_c \frac{e^{i-z}}{(z-1)^2}dz = \frac{)n(}{e}$

3.2 Calculation of Residues

If $z = z_0$ is a pole of order K, there is a formula for b_n given as

$$b_n = \frac{\lim}{z \to z_0}\frac{1}{(n-1)!}\frac{d^{ii-i}}{dz^{n-1}}\{(z-z_0)^n f(z)\}\text{………………..}(5)$$

If $\eta = 1$ (simple pole), the result is given as

$$b_1 = \frac{\lim}{z \to z_0}(z-z_0)f(z)$$

$z \to z_0$

Which is a special case $-f$ (5) with $\eta = 1$ if one defines $0! = 1$.

Example

For each of the following functions, determine the poles and the residues at the poles.

(a) $\frac{2z+1}{z^2-z-2}$ (b) $\left(\frac{z+1}{z-1}\right)^2$

Solution:

(a) $\frac{2z+1}{z^2-z-2}=\frac{2z+1}{(z+1)(z-2)}$ ……the function has two poles at $z=-1$ and $z=2$ both of under 1.

Residue at $z=-1$,

$$\lim_{z\to -1}(z+1)f(z)=\lim_{z\to -1}\frac{(z+1)(2z+1)}{(z+1)(z-2)}$$
$$=\lim_{z\to -1}\frac{2z+1}{z-2}=\frac{1}{3}$$

Residue at z = 2,

$$\lim_{z=2}\frac{(z-2)(2z+1)}{(z+1)(z-2)}=\lim_{z\to 2}\frac{2z+1}{z+1}=\frac{5}{3}$$

(b) z = 1 is a pole of order 2.
Residue at z = 1 is

$$\lim_{z\to 1}\frac{y}{dz}\left\{(z-1)^2(z+1)^2/(z-1)^2\right\}$$
$$\lim_{z\to 1}\frac{d}{dz}(z+1)^2=\lim_{z\to 1}2(z+1)=4.$$

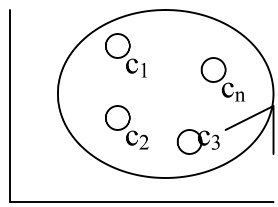
3.3 Residue Theorem

Theorem: Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2,\ldots, z_n$ interior to C. If $B_1, B_2,\ldots, B_n$ denote the residues of f at these points respectively, then

$$\int_c f(z)dz = 2\pi i\,(B_1+B_2+\ldots\ldots+B_n)\ \ldots\ldots(1)$$

Proof

Let the singular points $zj\ (j=1,2,\ldots.n)$ be centers of positively oriented circles Cj which are interior to C and are so small that no two of the circles have points in common.



The circles Cj together the simple closed contour C form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain. Hence, according to the extension of the Cauchy-Goursat theorem to such regions.

$$\int_c f(z)dz = \int_{c_1} f(z)dz - \int_{c_2} f(z)dz - ....- \int_{c_n} f(z)dz = 0$$

This reduces to equation (1) because

$$\int_{c_1} f(z)dz = 2\overline{\pi} iBj \qquad (1, 2, ...n)$$

And the proof is complete.

Example: Let us use the theorem to evaluate

$$\int_c \frac{5z-2}{z(z-1)}\, dz$$

Where C is the circle $|z| = 2$, described counter clockwise. The integrated has the two singularities $z = 0$ and $z = 1$, both of which are interior to C. We can find the residues B_1 at $z = 0$ and B_2 at $z = 1$ with the aid of the maclurin series.

$$\frac{1}{1-z} = 1 + z + z\tfrac{2}{1}............ \qquad (|z| < 1)$$

We first write the Laurent expansion

$$\frac{5z-2}{z(z-1)} = \left(\frac{5z-1}{z}\right)\left(\frac{-1}{1-z}\right) = \left(5 - \frac{2}{z}\right)\left(-1 - z - z^2..........\right)$$

$$= \frac{2}{z} - 3 - 3z.............................. \qquad (0 < |z| < 1)$$

Of the integrand and conclude that $B_1 = 2$. Next, we observe that

$$\frac{5z-2}{z(z-1)} = \left[\frac{5(z-1)+3}{z}\right]\left[\frac{1}{1+(z-1)}\right]$$

$$= \left(5 + \frac{3}{z-1}\right)\left(1-(z-1)\right) + (z-1)^2........$$

When $0 < |z-1| < 1$. The coefficient of $1/(z-1)$ in the Laurent expansion which is valid for $0 < |z-1| < 1$ is therefore 3

Thus $B_2 = 3$, and

$$\int_c \frac{5z-2}{z(z-1)}dz = 2\overline{\pi}i(B_1 + B_2) = 10\overline{\pi i}.$$

An alternative and simple way of solving the problem is to write the integrand as the sum of its partial fractions. Then

$$\int_c \frac{5z-2}{z(z-1)}dz = \int_c \frac{2}{z}dz + \int_c \frac{3}{z-1}dz = 4\pi i + 6\pi i = 10\pi i$$

4.0 CONCLUSION

The residue method learnt in this unit allows us to handle integration with ease. You are required to master this method very well.

5.0 SUMMARY

Recall that we started this unit by defining the residue theorem which is now recalled for your understanding:

Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points z_1, z_2,…, z_n interior to C. If B_1, B_2,…,B_n denote the residues of f at these points respectively, then

$$\int_c f(z)dz = 2\pi i\,(B_1 + B_2 + \dots\dots\dots + B_n).$$

This theorem form the basis for solves complex integration. You may wish to answer the following tutor-marked assignment question.

6.0 TUTOR-MARKED ASSIGNMENT

1. Evaluate the integral

$$\int_c \frac{7z-5}{z-1}dz$$

2. Evaluate

$$\int \frac{z+2}{z^2-5z+6}dz$$

3. $\int_c \frac{5z-2}{z^2(z-1)}dz$

7.0 REFERENCE/FURTHER READING

Hildebrand, Francis B. (2014). *Advanced Calculus for Application*. 6th Edition.

UNIT 4 COMPLEX INTEGRATION

CONTENTS

1.0 INTRODUCTION

This unit will examine complex integration. The theorem on line integral, such as green's theorem will also be examined.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define integration on complex variables;
- define the complex form of green's Theorem;
- describe Cauchy-Goursat theorem;
- describe Cauchy integral; and
- solve related problems on complex integrations.

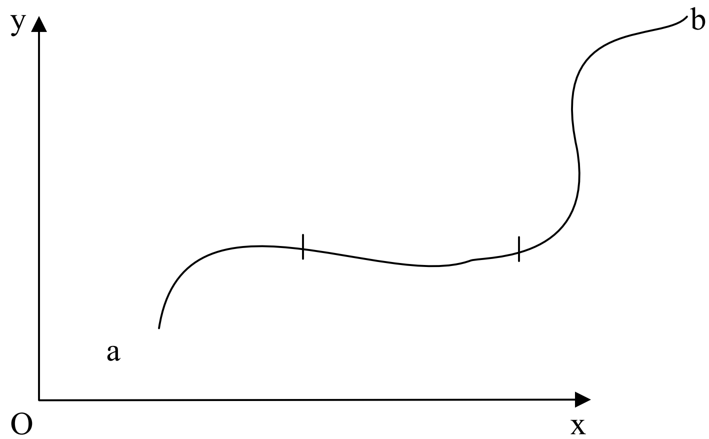
3.0 MAIN CONTENT

3.1 Curves

If $\phi(t)$ and $\psi(t)$ are real functions of the real variable t assumed continuous in $t_1 \leq t \leq t_2$, the parametric equations

$Z = x + iy = \phi(t) + i\psi(t) = Z(t) \qquad t_1 \leq t \leq t_2$

Define a continuous curve or arc in the Z-plane joining points $a = Z(t_1)$ and $b = Z(t_2)$ as shown below



If t_1 µ t_2 while $Z(t_1) = Z(t_2)$, i.e. $a = b$, the end point is coincide and the curve is said to be closed. A close curve which does not intersect itself anywhere is called a simple closed curve.

If $\phi(t)$ and $\psi(t)$ have its derivations in $t_1 \le t \le t_2$, the curve is often called a smooth curve or arc. A curve which is composed of a finite number of smooth arcs is called a piecewise or sectionally smooth curve or sometimes a contour. For example, the boundary of a square of a piecewise smooth curve or contour.

3.2 Simply and Multiply Connected Regions

A region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply connected is called multiply-connected e.g. $|z| < 2$. $|<||z| < 2$.

3.3 Complex Line Integrals

Suppose that the equation

$z = z(t) \qquad (a \le i \le b)$………………………………………….. (4.1.1)

Represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. Let the function $f(z) = \mu(x, y) + iv(x, y)$ be piecewise continuous on C. If $z(t) = \mu(t) + iy(t)$ the function

$f[z(t)] = \mu[x(t), y(t)] + i\,v[x(t), y(t)]$ is piecewise continuous on the interval $a \le t \le b$. We define the line integral or contour integral, of f along C as follows:

$$\int_c f(z)dz = \int_a^b [z(t)] \; z^1(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (4.1.2)$$

Note that since C is a contour, $z(t)$ is also piecewise continuous on the integral $a \leq t \leq b$, and so the existence of integral (4.1.2) is ensured.

The integral on the right-hand side in equation (4.1.2) is the product of the complex-valued functions.

$\mu[x(t), y(t)] + i \; v\,[n(t), y(t)], \;\; n^1(t) + y^1(t).$
Of the real variable t. Thus

$$\int_c f(z)dz = \int_a^b \left(\mu x^1 - vy^1\right)dt + i \int_a^b \left(vn^1 + uy^1\right)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (4.1.3)$$

In terms of line integrals of real-valued functions of two real variables, then

$$\int_c f(z)dz = \int_c \mu dx - vdy + i \int vdu + udy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.1.4)$$

Example: Find the value of the integral

$$I_1 = \int_{c_1} z^2 dz$$

Where C_1 is the line segment from $z = 0$ to $z = 2 + i$

Proof
Points of C_1 lie on the line $y = x/2$ or $x = 2y$. If the coordinate y is used as the parameter, a parametric equation for C_1 μ

$$z = 2y + i\,y \qquad (0 \leq y \leq 1)$$

Also, in C_1 the integral z^2 becomes

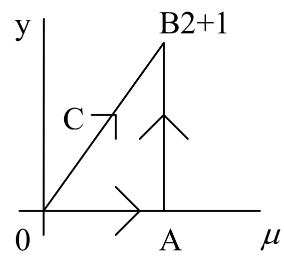
$$z^2 = (2y + iy)^2 = 3y^2 + i4y^2$$

Therefore,

$$I_1 = \int_0^1 \left(3y^2 + i4y^2\right)\,(2 + i)\;dy$$

$$= (3 + 4i)(2 + i)\int_0^1 y^2 dy = \tfrac{2}{3} + \tfrac{11}{3}\,i$$

Example: Let C_2 denote the contour $0AB$ shown below



Evaluate

$$I_2 = \int_{c_2} z^2 dz$$

Solution:

$$I_2 = \int_{c_2} z^2 dz = \int_{0A} z^2 dz + \int_{AB} z^2 dz$$

The parametric equation for path 0A is $z = n + i0(1 \le x \le 2)$ and for the path AB one can write $Z = 2 + iy(0 \le y \le 1)$.

Hence

$$I_2 = \int_0^2 x^2 dx + \int_0^1 (2+iy)^2 idy.$$

$$= \int_0^2 x^2 dx + 2\left[\int_0^1 (4 - y^2)dy + 4i\int_o^1 ydy\right]$$

$$= \frac{2}{3} + \frac{11}{3}\ i$$

Green's Theorem in the Plane

Let $P(x\ , y)$ and $Q(x, y)$ be its and have its partial derivatives in a region R and on its bounding C. Green's theorem states that

$$\oint_c Pdx + Qdy = \int_R \int (Q_x - P_y)dndy$$

The theorem is valid for both simple and multiple connected regions.

3.4 Complex Form of Green's Theorem

Let $F(z, \bar{z})$ be its and have its derivations in a region R and on its bounding C, where $z = x + iy$, $\bar{z} = x - iy$ are complex conjugate coordinates. The Green's theorem can be written in the complex form as

$\oint_c F(z,\bar{z})dz = 2i\iint \frac{\partial F}{\partial \bar{z}}\, dA$ where dA represents the element of area $dndy$

Proof

Let $F(z,z) = P(x) + iQ(x,y)$. Then using Green's theorem, we have

$$\oint_c F(z_1\bar{z})dz = \oint_c (P+iQ)(x,y)dz \qquad = \oint_c Pdn-,Qdy + i\oint_c Qdn + P\,dy$$

$$= -\oiint_R \left(\frac{\partial Q}{\partial^R} + \frac{\partial P}{\partial y}\right) dndy + i\oiint_R \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dxdy$$

$$= i\iint_R \left[\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right] dx\,dy$$

$$= 2i\iint_R \frac{\partial F}{\partial \bar{Z}}\, dndy$$

Example: Evaluate the integral

$I = \int \bar{z}dz$

Where

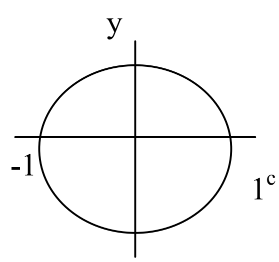
(i) The path of integration C is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$.

(ii) Same points but along the lower semi circle C.

Solution:

(i) The parametric representation $z = e^{i\alpha}$ $(0 \leq \notin \leq i)$ and since $d\left(e^{iQ}\right)/d\notin = ie^{i\notin}$

$$I = \int_c \bar{z}dz = -\int_0^{\bar{\wedge}} e^{-i\notin} i\, e^{i\notin} d0 = -\pi i$$



(ii) $I = \int \bar{z}dz = \int_{\wedge}^{2\bar{\wedge}} e^{-1Q} i\, e^{i\notin}\, d\phi = \pi i$

Example: Evaluate $\int \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by (a) $z = t^2 + it$
(b) the line from $z = 0$ and $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$

Solution:

(a) The given integral equal,

$$\int_c (n - iy)(dn + idy) = \int_c ndk + ydy + i \int_c ndy - ydn$$

The parametric equations of C are $n = t^2, y = t$ from $t = 0$ to $t = 2$
Then the line integral equal

$$\int_{t=0}^{2} (t^2)(2tdt) + (t)(dt) + i\int_{t=0}^{2} (t^2)(dt) - (t)(dt - dt)$$

$$= \int_0^2 (2i^3 + t)dt + i\int_0^2 (-t^2)dt = 10 - \frac{8i}{3}$$

(b) $\int_c (x - iy)(dx + idy) = \int_c xdx + ydy + i\int_c ndy - ydx$

The line from $Z = 0 \; to \; Z = 2i$ is the same as $(0,0)$ to $(0,2)$ for which $x = 0, \; dn = 0$ and the line integral equals.

$$\int_{y=0}^{4} (Q)(0) + ydy + i\int_{y=0}^{2} (0)dy - y(0) = \int_{y=0}^{2} ydy = 2 .$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from $(0,2)$ to $(4,2)$ for which $y = 2, dy = 0$ and the line integral equals

$$\int_0^4 xdx + 2 \bullet 0 + i\int_{x=0}^{4} n \bullet 0 - 2dn = \int_0^4 xdx + i\int_0^4 \frac{-2xdx}{8i} = 8 - 8i$$

Then the requires value $= 2 + (8 - 8i) = 10$.

3.4 Cauchy-Goursat Theorem

Suppose that two real-valued function $P(n, y)$ and $Q(n, y)$ together with their partial derivatives of the first order, are continuous throughout a closed region R consisting of points interior to and on a simple closed contour C in the ny plane. By Green's theorem, for line integrals,

$$\int_c Pdn + Qdy = \iint_R (\phi_x - P_y)dndy.$$

Consider a function

$f(z) = u(x, y) + i \; v \; (x, y)$

Which is analytic throughout such a region R in the ny, or Z, plane, the line integral of f along C can be written

$$\int_c f(z)dz = \int_c ndn - vdy + i \int_c vdx + udy \text{ ....(1)}$$

Since f is its in R, the functions u and v are also its theorem i and if the derivative f^1 of f is its in R, so are the first order partial derivatives of u and v. By Green's theorem, (1) could be written as

$\int_c f(z)dz = \iint_R (-v_x - u_y)dxdy + i \iint_R (u_x - v_y)dxdy$ ……………………….(2)

But in view of the Cauchy-Goursat equations
$U_x = V_y, U_y = -V_x$
The integrals of these two double integral are zero throughout R. So

Theorem: If f is analytic in R and f^1 is continuous then, $\int_c f(z)dz = 0$.
This is known as Cauchy theorem.

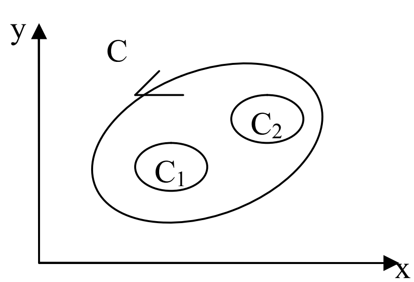
Goursat proved that the condition of continuity of f^1 in the above Cauchy theorem can be omitted.

Theorem: (Cauchy-Goursat theorem)
If a function ***f*** is analytic at all points interior to and in a simple closed contour C, then $\int_c f(z)dz = 0$.

Cauchy-Goursat theorem can also be modified for the …… B of a multiply connected domain.

Theorem: Let C be a simple closed contour and let Cj $(j = 1,2,....n)$ be a finite number of simple closed contours inside C such that the regions interior to each Cj have no points in common. Let R be the closed region consisting of all points within and on C except for points interior to each Cj. Let B and all the contours oriented boundary of R consisting of C and all the contours Cj, described in a direction such that the interior points of R lie to the left of B. Then, if f is analytic throughout R_1.

$\int_B f(z)dz = 0$

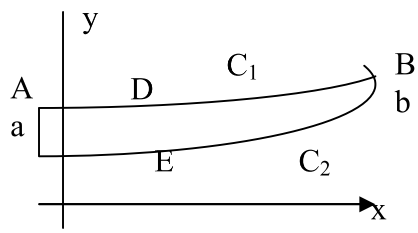


As a consequence of Cauchy's theorem, we have the following

Theorem: If $f(z)$ is analytic in a simply-connected region R, then $\int_a^b f(z)dz$ is independent of the path in R joining any two points a and b in R.

Proof

Consider the figure below



By Cauchy's theorem

$$\int_{ADBCA} f(z)dz = 0$$

Or $\int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0$

Hence

$$\int_{ADB} f(z)dz = -\int_{BEA} f(z)dz = \int_{AEB} f(z)dz$$

Thus

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_a^b f(z)dz$$

This yields the required result.

Example: If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1, 1) and (2, 3), show that
$\int_c (12z^2 - 4iz)dz$ is independent of the path joining (1, 1) and (2, 3)

Solution:

A (1, 1) $\rightarrow$ B (2, 1) $\rightarrow$ C (2, 3)

Along A (1, 1) to B (2, 1), y = 1, dy = 0. So that $z = x4i$ and $dz = dx$. Then

$$\int_{x-1}^{2}\{12(x4i)^2 - 4i(x+i)\}dx = 20 + 30i$$

Along B (2, 1) to C (2, 3), $x = 2,\ dx = 0$ so that $z = 2 + iy$ and $dz = idy$. Then

$$\int_{y=1}^{3}\{12(2+iy)^2 - 4i(2+iy)\}idy = -176 + 8i$$

$y = 1$

So that

$$\int_c (12z^2 - 41z)dz = 20 + 30i - 176 + 8i = -156 + 38i$$

The given integral equals

$$\int_{1+i}^{2+3i}(12z^2 - 4iz)dz = (4z^3 - 2iz^2)\Big|_{1+i}^{2+3i} = -156 + 38i$$

Morera's Theorem

Let $f(z)$ be continuous in a simply connected region R and suppose that

$$\oint_c f^1(z)dz = 0$$

Around every simple closed curve C in R. Then $f(z)$ is analytic in R.

This theorem is called the converse of Cauchy's theorem and it can be extended to multiply-connected regions.

Indefinite Integrals (Anti-derivatives)

Let $f(z)$ be a function which is continuous throughout a domain D, and suppose that there is an analytic function F such that $F^1(z) = f(z)$ at each point in D. The function F is said to be an anti derivative of f in the domain D.

Cauchy Integral Formula

Theorem: Let f be analytic everywhere within and in a simple closed contour C taken in the positive sense. If Z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{Z - z_0}$$

This formula is called the Cauchy integral formula. It says that that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f in C.

When the Cauchy integral formula is written as

$$\int_c \frac{f(a)dz}{Z - z_0} = 2\bar{\pi i}\ f(z_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

It can be used to evaluate certain integrals along.

Simple closed contours

Example: Let C be the positively oriented circle $|z| =$ since the function $f(z) =^2 /(9 - z^2)$ is analytic within and in C and the point $Z_0 = -i$ is interior to C, then by Cauchy Integral formula

$$\int_c \frac{zdz}{(9 - z^2)(z + i)} = \int_c \frac{z/(9 - z^2)}{z - (-i)} = 2\pi c\left(\frac{-i}{10}\right) = \frac{\pi}{\delta}$$

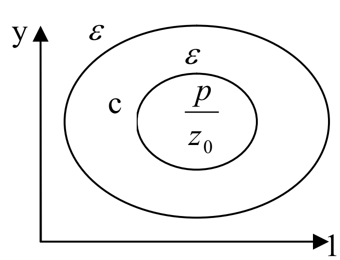
Proof

Since f is its at Z_0, there corresponds to any positive number ε , however small, a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| = \rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Observe that the function $f(z)/(z-z_0)$ is analytic at all points within and in C except at the point z_0. Hence, by Cauchy-Goursat theorem for multiply connected domain, it's integral around the oriented boundary of the region between C and C_0 has value zero.

$$\int_c \frac{f(z)dz}{Z - z_0} - \int_{C_0} \frac{f(z)dz}{Z - z_0} = 0$$



That is

$$\int_c \frac{f(z)dz}{Z-Z_0} = \int_{C_0} \frac{f(z)dz}{Z-Z_0}$$

This allows us to write

$$\int_C \frac{f(z)dz}{Z-Z_0} - f(z_0)\int_{C_0} \frac{dz}{Z-Z_0} = \int_{C_0} \frac{f(z)-f(z_0)}{Z-Z_0}dz \text{ ....(2)}$$

$$\int_{C_0} \frac{dz}{Z-Z_0} = 2\pi i$$

And so equation (Z) becomes

$$\int_C \frac{f(z)dz}{Z-Z_0} - 2\pi i\ f(z_0) = \int_{C_0} \frac{f(z)-f(z_0)}{Z-Z_0}dz \text{ ....(3)}$$

By (1) and noting that the length of C_0 is $2\overline{\wedge}\rho$, by properties of integrals

$$\left|\int_{C_0} \frac{f(z)-f(z_0)}{Z-Z_0}dz\right| < \frac{\varepsilon}{\rho}2\overline{\pi}\rho = 2\pi\varepsilon$$

In view of (3) then

$$\left|\int_C \frac{f(z)dz}{Z-Z_0} - 2\pi i\ f(z_0)\right| < 2\overline{\pi}\varepsilon .$$

Since the left hand side of this inequality is a non negative constant which is less than an arbitrary small positive number, it must be equal to zero. Hence, equation for it valid and the theorem is proved.

Cauchy's integral formula can also be extended to a multiply connected region. With the understanding that $f\overset{(v)}{\underset{z}{}}$ denotes $f(z)$ and that 0! = 1, we can use mathematical induction to verify that

$$f^{(n)}(z_0) = \frac{n!}{2\overline{\pi}i}\int_C \frac{f(z)dz}{(Z-Z_0)^{n+1}}(n = v,1,2.)$$

When n = 0, this is just the Cauchy integral formula stated earlier.

Example: Find the value of $\oint_C \frac{Sin^6 Z}{\left(Z - \bar{\pi}/6\right)^3} dz$

Where C is a circle $|z| = 1$

Solution:

$$\int_C \frac{Sin^6 z}{\left(Z - \pi/6\right)^3} dz = \frac{2\pi i\, f^2\left(Sin^6 \pi/6\right)}{2^1}$$

$$= \frac{6x2\bar{\pi} i}{2}\left[5\, Sin^4 \pi/6\ \ Cos^2 \pi/6 - Sin^6 \pi/6\right]$$

$$= 2\,1\pi i/16$$

Other Important Theorems

1. **Cauchy's inequality**
   If $f(z)$ is analytic inside and on a circle C of radius r and centre at $z \neq a$, then $\left|f^{(n)}(a)\right| \leq \frac{\mathrm{M} \bullet n!}{r^n}$ $n = 0, 1, 2, ...$
   Where M is a constant such that $|f(z)| < M$ on C, i.e. M is an upper bound of $|f(z)|$ on C.

2. **Lowville's Theorem**
   Suppose that for all Z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded, i.e. $|f(z)| < M$ for some constant M, then $f(z)$ must be a constant

3. **Fundamental Theorem of Algebra**
   Every polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \quad a_n Z^n = 0$ with degree $\pi \geq 1$, and $a_n \neq 0$ has at least one root.

4. **Maximum Modulus Theorem**
   If $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant, then the maximum values of $|f(z)|$ occurs on C.

SELF - ASSESSMENT EXERCISES

1. Evaluate $\int_{(0,1)}^{(2,5)} (3x + y)dx + (xy - x)dy$ along
   (a) the curve $y = x^2 + 1$
   (b) the straight line joining (0, 1) and (2, 5)
   (c) the straight line from (0, 1) to (0, 5) and then (0, 5) to (2, 5)

2. Evaluate $\int_C \left(x^2 - iy^2\right)dz$

   (a) along the parabola…… $y = 4n^2$ from (1,4) to (2, 16)

   (b) straight line from (1, 1) to (1, 8) and then from (1, 8) to (2, 8).

3. Evaluate $\int_{-2+i}^{2-i} \left(3xy + iy^2\right)dz$

   (a) along the curve $x = 2t - 2\ i - y = 1 + t - t^2$

   (b) along the straight line joining $x = -2 + i$ and $z = 2 - i$

4. Evaluate

   (a) $\oint_C \frac{Sin\,\pi Z^2 + Cos\,\pi Z^2}{(Z-1)(Z-2)}dz$, where C is the circle $|Z| = 3$.

   (b) $\oint_C \frac{e^{2z}}{(Z+1)^4}dz$ where C is the circle $|Z|$=3

5. Evaluate $\oint_C \frac{Sin\,3z}{Z + \pi/2}dz$ if C is the circle $|Z| = 5$

4.0 CONCLUSION

The materials in this unit must be learnt properly because they will keep on re occurring as progress in the study of mathematics at higher level.

5.0 SUMMARY

We recap what we have learnt in this unit as follows:

You learnt about Cauchy-Goursat equations, Moreras Theorem and applied it to indefinite integrals. We also consider Cauchy integral formula

We considered some solved examples to illustrate the theory we have learnt in this unit. You may which to answer the following tutor-marked assignment.

6.0 TUTOR-MARKED ASSIGNMENT

1. Evaluate $\int_{-2+i}^{2-i} \left(3xy + iy^2\right)dz$

   (a) along the curve

   (b) along the straight line joining $x = -2 + i$ and $z = 2 - i$

2. Evaluate

(a) $\oint_C \frac{Sin\,\pi z^2 + Cos\,\pi z^2}{(z-1)(z-2)}dz$, where C is the circle $|z| = 3$.

(b) $\oint_C \frac{e^{2z}}{(z+1)^4}dz$ where C is the circle $|z|=3$

3. Evaluate $\oint_C \frac{Sin\,3z}{z+\pi/2}dz$ if C is the circle $|z| = 5$

7.0 REFERENCE/FURTHER READING

Hildebrand, Francis B. (2014). *Advanced Calculus for Application*. 6th Edition.