MODULE 1 EXISTENCE AND UNIQUENESS OF SOLUTIONS

UNIT 1 ORDINARY DIFFERENTIAL EQUATION

CONTENTS

1.0 INTRODUCTION

In this unit, we shall study the theory of ordinary differential equations with a discussion on existence and uniqueness theorems which cover various types of equations. A differential equation is a functional equation where the unknown function or functions are present as derivatives with respect to a single variable in the case of an ordinary differential equation. The order of the highest derivative is called the order of the equation. Derivatives in a differential equation can occur in various ways and we do not admit equations where the unknown is subjected to other operations than algebraic and differential equations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- classify various types of differential equation; and
- answer correctly exercises on differential equations.

3.0 MAIN CONTENT

3.1 Definitions and Examples

A differential equation is a functional equation where the unknown function or functions are present as derivatives with respect to single variables in the case of an ordinary differential equation. Consider the following six examples of functional equations involving derivatives. Some are bona fide equations while some are not:

Example (1): $f'(x) = f(x)$
Example (2): $f'(x) = f(x+1)$
Example (3): $f'(x) = a_0(x) + a_{(1}(x) f(x) + a_2(x)[f(x)]^2$
Example (4): $f''(x) = 6x + [f(x)]^2$
Example (5): $f'(x) = \int_0^x \{1 + [f(s)]^2\}^{1/2} ds$
Example (6): $f(x) = \int_0^1 \{[f'(s)]^2 + [f(x)]^2\}^{1/2} ds$

Examples 1 and 3 are ordinary and first order differential equations, while example 2 is a different differential equation, not a differential equation in the usual sense. Example 4 is a second order differential equation. Example 5 is not a differential equation as it stands but on differentiating will yield
$f''(x) = \{1 + [f(x)]^2\}^{1/2}$ which is a second order differential equation equivalent to example 4. Finally, example 6 is not a differential equation and is not reducible to such an equation by elementary means.

The normal form of a first order differential equation is given as
$y' = F(x, y)$ . . . (1)

In the simplest case, x and y are real variables and $F(x, y)$ is a function on R^2 to R^1. We can also allow x and y to be complex variables and F to be a function on C^2 to C^1.

We can also let
$y = (y_1, y_2, y_3, y_4, ..., y_n)$ and F= $(F_1, F_2, F_3, ..., F_n)$ ……. (2)

Where y and F are functions on R^{n+1} to R^1. We then define the derivative of a vector as the vector of the derivatives:
$y' = (y_1', y_2', y_3', ..., y_n')$ …… (3)

With this notation, equation (1) becomes a condensed convenient way of writing a system of first order differential equations:
$y_j'(x) = F_j(x, y_1, y_2, y_3, ..., y_n), j = 1,2,3, ..., n$ . . . (4)

Conversely, every such system can be writing as a first order vector differential equation. The generalisation has the advantage of covering **nth** order equations. To convert nth order differential equation in y to a first order vector equation in y, we set
$y = (y, y', y'', ..., y^{(n-1)})$ …. (5)

We can consider differential equations in more general spaces than the Euclidean. Here, the interpretation of the derivatives becomes a matter of concern, and convergence questions also arise if the space is of infinite dimension.

Differential equations normally have infinite number of solutions. In order to find a particular one we have to impose some special conditions on the solution, usually an initial condition. The intent of an existence theorem is to show that there exists a function which satisfies the equation in some neighborhood of point (x_0, y_0). A uniqueness theorem asserts that there is only one such function. We can, however, assert the existence of solution under much more general conditions than those which guarantee uniqueness. This is beyond the scope of this course.

4.0 ACTIVITY 1

Solve the differential equation $y'' - 3y' + 2y = 0$

5.0 CONCLUSION

We have examined differential equations in a general setting in this unit. This unit is important to the understanding of other units that would follow subsequently.

6.0 SUMMARY

In this unit, we have a general introduction to various forms of differential equations. This unit must be read carefully before proceeding to the other units.

7.0 TUTOR-MARKED ASSIGNMENT

i. If $f(x)$ satisfies the integral equation

$$f(x) = y_0 + \int_{x_0}^{x_1} F[s, f(s)]ds,$$

Find a differential satisfied by $f(x)$. What initial condition does $f(x)$ satisfy?

ii. Transform $f(x) = \int_0^x [f(s)]^2 ds$ into differential equation. Here $f(x) = 0$ is obviously a solution. Are there other solutions of the functional equation?

iii. The functional equation $f(x) = 1 + \int_0^x f(s)ds$ implies that f satisfies a differential equation. Find the latter and find the common solution.

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (1989). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

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UNIT 2 THE FIXED POINT METHOD

CONTENTS

1.0 INTRODUCTION

In this unit, we shall use a topological method based on the contraction fixed point theorem. To apply this theorem successfully we have to replace the differential equation by an equivalent integral equation that can be used to define a contraction operator on a suitably chosen metric space.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply the contraction fixed point theorem;
- determine the existence of solutions for a given differential equation; and
- solve correctly the tutor-marked assignment that follows.

3.0 MAIN CONTENT

3.1 The Fixed Point Method

Consider the following differential equation defined by

$$f'(x) = F[x, f(x)], f(x_0) = y_0 \qquad \ldots (1)$$

Here $F = (F_1, F_2, ..., F_n)$ is a vector valued function defined and continuous in $B: |x - x_0| < a, \quad \|y - y_0\| < b$

We may define the norm on R^n as follows:

$$\left[\sum_1^n (y_j - y_{j0})\right]^{1/2} \quad \text{or} \quad \max|y_j - y_{j0}|$$

We impose two further conditions on F :

$$\|F(x,y)\| \le M \quad \ldots (2)$$

$$\|F(x,y_1) - F(x,y_2)\| \le K\|y_1 - y_2\| \quad \ldots (3)$$

Conditions (2) and (3) are called boundedness and Lipschitz conditions respectively.

We now replace the vector differential equation by a vector integral equation defined as:

$$f(x) = y_0 + \int_{x_0}^{x} F[s, f(s)]ds \quad \ldots (4)$$

We again impose the following property which follows from the definitions of integrals by Riemann as:

$$\left\|\int_{x_0}^{x} Fds\right\| \le \int_{x_0}^{x} \|F\|ds, x_0 < x \quad \ldots (5)$$

Theorem (1): Under the stated assumptions on F, the equation (1) has a unique solution defined in the interval ($x_0 - r, x_0 + r)$ where

$$r < \min\left(a, \frac{b}{M}, \frac{1}{K}\right)$$

Proof: We consider the space N of all functions $g(x)$ on R^1 *to* R^n continuous in x $(x_0 - r, x_0 + r)$ such that $g(x_0) = y_0$ and $\|g - y_0\|_N \le b$ where

$\|g - y_0\|_N = \sup_x \|g(x) - y_0\|.$. For such, a $g(x)$ the function $F[x, g(x)]$ exists and is continuous. Furthermore, its N-norm does not exceed M. We now define the transformation:

$$T : g(x) \to y_0 + \int_{x_0}^{x} F[s, g(s)]ds, \quad -r < x - x_0 < r \quad \ldots (6)$$

Here, $T[g](x)$ is continuous, $T[g](x_0) = y_0$ and

$$\|T[g](x) - y_0\| < Mr < b \quad \ldots (7)$$

(by the choice of r). It follows that $T[g] \in N$. We next observe that

$$\|T[g_1](x) - T[g_2](x)\| = \left\|\int_{x_0}^{x} \{F[s, g_1(s)] - F[s, g_2(s)]\}ds\right\| < K\left|\int_{x_0}^{x} \|g_1(s) - g_2(s)\|ds\right|$$

This shows that

$$\|T[g_1]-T[g_2]\|_N \leq Kr\|g_1-g_2\|_N = k\|g_1-g_2\|_N$$

Where $Kr = k < 1$ by choice of r. Hence T is a contraction. This implies that there exist one and only one function $f(x) \in N$ such that

$f(x) = y_0 + \int_{x_0}^{x} F[s, f(s)]ds, f(x_0) = y_0$ is the unique solution of the differential equation (1) with the stated initial condition.

4.0 CONCLUSION

We have shown that we can apply the fixed point theorem to establish the existence of solution to the differential equation stated in (1). You are supposed to master the concept developed in this unit before proceeding to the next unit.

5.0 SUMMARY

The contraction fixed point theorem applied in this unit enables us to develop a unique solution to the differential equation stated in (1). It is one of the most powerful theorems in mathematical analysis. It can be extended to spaces of infinitely in many dimensions. However, this is beyond the scope of this unit.

6.0 TUTOR-MARKED ASSIGNMENT

Determine an interval $(x_0 - r, x_0 + r)$ where the existence of solution to the following differential equations is guaranteed:

i. $y' = y, y(0) = 1$

ii. $y' = y^3, y(0) = 2$

iii. $y' = xy + y^2, y(0) = 0$

iv. $y_1' = y_1 + y_2, y_1(0) = -1, y_2(0) = 1$

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (1989). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

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UNIT 3 THE METHOD OF SUCCESSIVE APPROXIMATIONS

CONTENTS

1.0 INTRODUCTION

The method of successive approximations is a refinement of the old device of trial and error. What has been added is control of the limiting process. We know how often the process must be repeated to bring the result with the desired limit of tolerance. The method of trial and error can be traced back to Isaac Newton who was the first to be concerned with approximate solution of algebraic equation. An infinite iteration process for the positive solution of the transcendental equation defined as:

$$x = \alpha \arctan x, \quad 1 < \alpha \quad \ldots\ldots.. \qquad \text{(A)}$$

was given by Joseph Fourier in his *Theorie Analytique de la Chaleur* (1822). Fourier's argument is geometrical and highly intuitive. It is not difficult to give a strict analytic convergence proof.

The method of successive approximation was given by Emile Picard for differential equation in 1891. This method soon became the standard method for proving existence and uniqueness theorems for all sorts of functional equations.

2.0 OBJECTIVES

At end of this unit, you should be able to:

- apply the method of successive approximation to determine existence and uniqueness of solution to differential equation.

3.0 MAIN CONTENT

3.1 The Method of Successive Approximations

Let us consider a vector differential equation defined by

$$y' = F(x, y), y(x_0) = y_0 . \qquad (1)$$

$F(x,y)$ is defined and continuous in:

$$B: |x - x_0| < a \quad \|y - y_0\| < b\,, \quad \|F(x,y)\| \le M \qquad (2)$$

$$\|F(x,y_1) \; - F(x,y_2)\| \le K\|y_1 - y_2\| \qquad (3)$$

We shall state the following theorem:

Theorem (1): There exists a unique function $f(x), on, R^1 to, R^n$ defined for $|x - x_0| < r$, where

$$r < \min\left(a, \frac{b}{M}\right) \qquad (4)$$

Proof: We replace the differential equation with the initial by the equivalent integral equation:

$$f(x) = y_0 + \int_{x_0}^{x} F[s, f(s)]ds \qquad (5)$$

$$f_0(x) = y_0$$

Now define

$$f_m(x) = y_0 + \int_{x_0}^{x} F[s, f_{m-1}(s)]ds, m = 1,2,3,.... \qquad (6)$$

For these functions to be well defined, we restrict x to the interval $(x_0 - r, x_0 + r)$. Suppose it is known that for some value of $m,$ the function $f_{m-1}(x)$ is well defined in this interval. It is obvious that $f_{m-1}(x) = y_0,$ but the induction hypothesis must also include that $f_{m-1}(x)$ is continuous and $\|f_{m-1}(s) - y_0\| < b.$ We then see that $F[s, f_{m-1}(s)]$ is well defined and continuous. Furthermore:

$$\|F[s, f_{m-1}(s)]\| \le M\,,$$

Hence

$\int_{x_0}^{x} F[s, f_{m-1}(s)]ds$, exist as a continuous function of x and its norm does not exceed $M|x - x_0| < Mr < b$ by the choice of $r.$

This implies that $f_m(x)$ is also continuous and satisfies $f_m(x_0) = y_0, \|f_m(x) - y_0\| < b$

It follows that the approximation are well defined for all $m.$ To prove the existence of $\lim f_m(x),$ we resort to the Lipschitz condition. We have

$$\| f_m(x) - f_{m-1}(x) \| = \left\| \int_{x_0}^{x} \{F[s, f_{m-1}(s)] - F[s, f_{m-2}(s)]\} ds \right\| \leq K \left| \int_{x_0}^{x} \| f_{m-1}(s) - f_{m-2}(s) \| ds \right|$$

We know that for some m we have the estimate

$$\| f_{m-1}(s) - f_{m-2}(s) \| \leq \frac{K^{m-2}}{(m-1)!} M |s - x_0|^{m-1}, |s - x_0| < r \qquad (7)$$

This estimate is certainly very true for $m = 2.$ We then get

$\| f_m(x) - f_{m-1}(x) \| \leq \frac{K^{m-1}}{(m-1)!} M \left| \int_{x_0}^{x} |s - x_0|^{m-1} ds \right| = \frac{K^{m-1}}{m!} M |x - x_0|^m$. Therefore the estimate is true for all m

Hence the series

$$f_0(x) + \sum_{n=1}^{\infty} [f_n(x) - f_{n-1}(x)] \qquad (8)$$

Whose partial sum is $f_m(x),$ converges in norm for $|x - x_0| < r$ uniformly in $x.$ Hence, the sum, $f(x),$ is a continuous function on $R^1 \longrightarrow R^n$.

The strong uniform convergence of the vector series (8) obviously implies the absolute and uniform convergence of the n component series to continuous functions on R^1, to, R^n. The estimate (7) obviously implies that

$$\| f(x) - f_m(x) \| \leq \frac{K^m}{m!} M \exp(K|x - x_0|)|x - x_0|^m \ldots\ldots \qquad (9)$$

It is an easy matter to observe that if $|x - x_0|$ is not large, $f_m(x)$ converges rapidly to its limit $f(x)$. Therefore, from the uniform convergence of $f_m(x)$ to $f(x)$ it follows that $F[s, f_{m-1}(s)]$ converges uniformly to $F[s, f(s)]$ and

$\int_{x_0}^{x} F[s, f_{m-1}(s)] ds \to \int_{x_0}^{x} F[s, f(s)] ds$ uniformly in $x.$ From (6) it follows that $f(x)$ satisfies (5) and consequently, the differential equation and the initial condition. That this is the only solution also follows from the Lipschitz condition. So to prove uniqueness we may suppose that $g(x)$ is a solution defined in some interval $(x_0 - r_1, x_0 + r_1).$ Then

$g(x) = y_0 + \int_{x_0}^{x} F[s, g(s)] ds$, and if $|x - x_0| < \min(r, r_1)$ we have

$$\| f(x) - g(x) \| = \left\| \int_{x_0}^{x} \{F[s, f(s)] - F[s, g(s)]\} ds \right\| \leq K \left| \int_{x_0}^{x} \| f(s) - g(s) \| ds \right|$$

Set $h(x) = \|h(x) - g(x)\|$, then $h(x)$ is a continuous non-negative function that satisfies, $0 \le h(x) \le K\left|\int_{x_0}^{x} h(s)ds\right|$. Hence $h(x)$ is identically 0. Therefore, $f(x)$ is the only solution of (1) with $f(x_0) = y_0$

4.0 CONCLUSION

Various questions arise when we want to use theorem (1) above. The first of these concerns the effective determination of a, b and M and the verification of the Lipschitz condition. We leave this for future considerations. We have justified the existence of solution to functional differential equations. We have also proved the uniqueness of this solution. You are required to read carefully before proceeding to the next unit.

5.0 SUMMARY

We have proved the existence of functional differential equations by successive approximation methods. Successive approximation method is essentially an iterative method that needs to be carefully designed to give a solution to the differential equation under consideration. Once the equivalent integral equation of the given differential equation is known, then it is just an easy matter to design the appropriate iterative scheme for the equation, which will eventually converge to the solution of the equation.

6.0 TUTOR-MARKED ASSIGNMENT

i. Solve $y' = y + x, y(0) = C,$ by method of successive approximation

ii. If $y(x)$ is a solution of

$$y'' - x^2 y = 0, y(0) = y_{01}, y'(0) = y_{02},$$

Show that

$$y(x) = y_{01} + y_{02}x + \int_0^x (x - s)s^2 y(s)ds.$$

Use the method of successive approximations to find $y(s)$ in the special case $y_{01} = 1, y_{02} = 0,$ Take $f(x) \equiv 1$

iii. The Thomas-Fermi equation defined by

$$x^{1/2} y'' = y^{3/2}$$

arises in nuclear physics. Show that it has a solution of the form Cx^{α}. Show also that it can be transformed into a system to which method of successive approximation can be applied so that its solution in some interval $[0, r]$ satisfies an initial condition of the form $y(0) = a > 0, y'(0) = b$.

7.0 REFERENCES/FURTHER READING

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