MODULE 2 SPECIAL FUNCTIONS

UNIT 1 SPECIAL FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall examine some special functions such as Beta function, Gamma function and Factorial function. These functions are of very useful mathematical importance in solving differential equations and other applied mathematics problems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define beta function, gamma function, and factorial notations; and
- apply these functions to solve mathematical problems.

3.0 MAIN CONTENT

3.1 Some Special Functions

Below are some of the special functions worthy of note.

3.1.1 Gamma Functions

One of the most important functions is the gamma function, written and defined by the integral

(1) $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \qquad \alpha > 0$

(More generally, if we consider also complex values, for those αt whose real part is positive). By integration by parts, we find

$\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^{\alpha} dt = -e^{-t} t^{\alpha} \|_0^\infty + \alpha \int_0^\infty e^{-t} t^{\beta-1} dt = \alpha\Gamma(\alpha)$

Thus we obtain the important functional <u>relation of the gamma function</u>

(2) $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

Let us suppose that the α $+ve$ integer, say, n. Then repeated application of (2) yields

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n(n-1)\Gamma(n-1)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &= n(n-1)\ldots\ldots\ldots\ldots3.2\Gamma(1)\end{aligned}$$

Now $\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t} \Big|_0^1 = 1$

(3) $\therefore \Gamma(n+1) = n! \ \therefore$

Hence gamma function can be regarded as a generalisation of the eliminating fractional function.

By repeated application of (2)

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{(\alpha)} = \frac{\Gamma(\alpha+2)}{(\alpha)(\alpha+1)} = \ldots\ldots\ldots\ldots \frac{\Gamma(\alpha+k+1)}{(\alpha)(\alpha+1)\ldots..(\alpha+k)}$$

Thus we obtain the relation

(4) $\Gamma(\alpha) = \lim \dfrac{\Gamma(\alpha+k+1)}{(\alpha)(\alpha+1)\ldots.(\alpha+k).} (\alpha \neq 0,-1,-2\ldots\ldots\ldots)$

Gauss defined Gamma function as follows

(6) $\Gamma(\alpha) = \lim\limits_{n\to\infty} \dfrac{n!n^{\alpha}}{\alpha(\alpha+1)\ldots\ldots..(\alpha+n)}$

Where

Problem I $,\alpha > 0$ and n is a $+ve$ integer, then

$$\Gamma(\alpha) = \lim_{n\to\infty}\int_0^n \left(1-t/n\right)^n t^{\alpha-1}dt.$$

Proof: Now consider the integral

$$\int_0^n \left(1-t/n\right)^n t^{\alpha-1}dt.$$

Substitute $t = nx$ in the integral, we obtain

$$\int_0^n \left(1-t/n\right)^n t^{\alpha-1}dt = n^\alpha \int_0^1 (1-x)^n x^{\alpha-1}dx$$

By integrating by parts gives the formula

$$\int_0^n (1-x)^n x^{\alpha-1}dx = \frac{n}{\alpha}\int_0^1 (1-x)^{n-1} x^\alpha dx$$

Repeating integration by parts, we get

$$\int_0^n (1-x)^n x^{\alpha-1}dx = \frac{n(n-1)(n-2).......1}{\alpha(\alpha+1)....(\alpha+n-1)}\int_0^1 x^{\alpha+n-1}dx$$

Thus

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{\alpha-1}dt = \frac{n!n^\alpha}{\alpha(\alpha+)...(\alpha+n-1)}\,\square\,\frac{1}{\alpha+n}$$

$$\therefore \lim_{n\to\infty}\int_0^n \left(1-t/n\right)^n t^{\alpha-1}dt = \lim_{n\to\infty}\frac{n!n^\alpha}{\alpha(\alpha+1)(\alpha+2)....(\alpha+n)} = \Gamma(\alpha)$$

Lemma1. If $0 \le \alpha < 1, \quad 1+\alpha \le \quad \exp(\alpha) \ \le (1-\alpha)^{-1}$, compare the three series.

$$(1+\alpha)^{-1} = 1+\alpha, \quad \exp(\alpha) = 1+\alpha+\sum_{N=2}^{\infty}\frac{\alpha^N}{n!}$$

$$(1+\alpha)^{-1=} = 1+\alpha+\sum_{N=2}^{\infty}\alpha^n$$

Lemma2. If $0 \le \alpha < 1,$, $(1-\alpha)^n \ge 1-\alpha^n$, for a position integer
Proof: For $n=1, \quad 1-\alpha = 1-\alpha,$ as derived.

Assume that

$$(1-\alpha)^\beta \ge 1-\beta\alpha,$$

Multiply each member by $1-\alpha,$ to obtain

$$(1-\alpha)^{\beta+1} \geq (1-\alpha)(1-\beta\alpha) = 1-(\beta+1)\alpha + \beta\alpha^2$$

So that

$$(1-\alpha)^{\beta+1} \geq 1-(\beta+1)\alpha = 1-(\beta+1)\alpha + \beta\alpha^2$$

Lemma 2. Follows by induction

Lemma 3. If $0 \leq t < n,\ n$ a positive integer

$$0 \leq e^{-t} - \left(1-\frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n!}$$

Proof: In Lemma I, put $\alpha = \frac{t}{n}$, we get

$$\left(1+\frac{t}{n}\right) \leq \mathrm{e}^{\frac{t}{n}} \leq \left(1-\frac{t}{n}\right)^{-1}$$

From which

(a) $\left(1+\frac{t}{n}\right)^n \leq \mathrm{e}^t \leq \left(1-\frac{t}{n}\right)^{-n}$

Or

$$\left(1+\frac{t}{n}\right)^{-n} \geq \mathrm{e}^{-t} \geq \left(1-\frac{t}{n}\right)^n$$

So that

$$e^{-t} - \left(1-\frac{t}{n}\right)^n \geq 0$$

$$\therefore e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\left[1-\left(1-\frac{t^2}{n^2}\right)^n\right]$$

But by (a)

$$e^{-t} \geq \left(1-\frac{t}{n}\right)^n$$

$$\therefore e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\left[1-\left(1-\frac{t^2}{n^2}\right)^n\right]$$

In Lemma 2, we have shown that

$$(1-\alpha)^n \geq 1-n\alpha.$$

$$\therefore \left(1-\frac{t^2}{n^2}\right)^n \geq 1-n\alpha$$

$$\therefore e^{-t} - \left(1-\frac{t}{n}\right)^n \leq \frac{e^{-t}t^2}{n}$$

Example 1: Show that the two definitions of gamma function are equivalent.

Proof: By using Gauss's definition, we proved that $\Gamma(z) = \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt$

Now

$$= \lim_{n\to\infty}\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1}dt + \int_0^\infty e^{-t}t^{z-1}dt$$

From the convergence of the integral

$$\int_0^\infty e^{-t}t^{z-1}dt = \Gamma(z)$$

It follows

$$= \lim_{n\to\infty}\int_0^n e^{-t} - t^{z-1}dt = 0$$

Hence

$$\int_0^\infty e^{-t}t^{z-1}dt - \Gamma(z) + \lim_{n\to\infty}\int_o^n \left[e^{-t} - \left(1-\frac{t}{n}\right) t^{z-1}dt\right.$$

Now

$\int_0^\infty e^{-t}t^{z-1}dt$ Converges, so $\int_0^n e^{-t}t^{z-1}dt$ is bounded.

Thus

$$\lim_{n\to\infty}\int_0^n \left[e^{t} - \left(1-\frac{t}{n}\right) t^{z-1}\right]dt = 0$$

$$\int_0^\infty e^{-t}t^{z-1}dt = \Gamma(z)$$

Example 2: Show that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{Sin\pi z} \qquad (z \neq 0. \neq 1, \neq 2,..........$$

Proof: using Gauss definition of gamma function

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)...........(z+n)} = \frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

$$\frac{1}{-z\Gamma(z)\Gamma(-z)} = z\prod_{s=1}^{\infty}\left(1-\frac{z^2}{s^2}\right)$$

$$= \frac{Sin\pi z}{\pi}$$

This implies that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{Sin\pi z}$$

Where

$$-z\Gamma(z)\Gamma(-z) = \Gamma(z)\Gamma(1-z)$$

Note if we put z = 1/2 , we get

$$\frac{1}{[\Gamma(1/2)]^2} = \frac{1}{\pi}$$

or

$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

Example 3: Show that

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2}) \qquad (2z = 0,-1,-2............)$$

Proof:

$$\frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)}$$

$$= \lim_{n\to\infty} \left(\frac{2^{2z} n! n^z n! n^{(z+\frac{1}{2})}}{z(z+1)...(z+n)(z+\frac{1}{2})(z+\frac{3}{2})\ldots(z+n+\frac{1}{2})}\right) \Big/ \frac{n! n^{2z}}{2z(2z+1)(2z+2)\ldots(2z+n)}$$

$$= \lim_{n\to\infty}\left[\frac{(n!)^2 2^{2n+1}}{(2n)!\sqrt{n}}\right]$$

The last quantity is independent of z and must be finite since the left side exists.

$$\therefore \frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} = A$$

Put z = ½

We have

$A = 2\sqrt{\pi}$

$$\therefore \Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})}{\sqrt{\pi}}$$

3.1.2 Beta-Function

We define Beta-function $B(p,q)$ by

(1) $\quad B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dx, R(c) > 0, R(q) > 0$

Another useful form of this function can be obtained by putting $t = Sin^2\theta,$ thus arriving at

(2) $\quad B(p,q) = 2\int_0^{\frac{\pi}{2}} Sin^{2p-1}\theta Cos^{2q-1}\theta d\theta, R(p) > 0, R(q) > 0$

Next we establish the relation between gamma and beta-functions

Example 1: If $R(p) > 0, R(q) > 0.$ Then

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Proof: $\Gamma(p)\Gamma(q)=\int_0^\infty e^{-t}t^{P-1}dt\int_0^\infty e^{-u}u^{q-1}du$

Substituting $t=x^2$ and $u=y^2$ it gives

$\Gamma(p)\Gamma(q)=4\int_0^\infty e^{-x^2}x^{2p-1}dx\int_0^\infty e^{-y^2}y^{2q-1}dy$

$\Gamma(p)\Gamma(q)=4\int_0^\infty\int_0^\infty \exp(-x^2-y^2)x^{2p-1}y^{2q-1}dxdy$

Next, turn to polar co-ordinate for the iterated integration over the first quadrant in xy-plan.

$\Gamma(p)\Gamma(q)=4\int_0^\infty\int_0^{\frac{\pi}{2}} \exp(-2^2)r^{2p+2q-2}Cos^{2p-1}\theta Sin^{2q-1}\theta rdrd\theta$

$2\int_0^\infty \exp(-r^2)r^{2p+2q-1}dr2\int_0^{\frac{\pi}{2}} Cos^{2p-1}\theta Sin^{2q-1}\theta d\theta$

Take $r^2=t$ and $\theta=\frac{1}{2}\pi-\theta,$ we obtain

$\Gamma(p)\Gamma(q)=\int_0^\infty e^{-t}t^{p+q-1}dt2\int_0^{\frac{\pi}{2}} Sin^{2p-1}\theta Cos^{2q-1}\theta d\theta$

$=r(p+q)B(p,q)$

$\therefore B(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$

3.1.3 Factorial Notations

(1) $(\alpha)_n=\prod_{k=1}^{n}(\alpha+k-1)$

$=\alpha(\alpha+1)......(\alpha+n-1)$

$(\alpha)_0=1,\alpha\neq 0$

The function $(\alpha)_n$ is called the factorial notation

Example 2: Show that

$$(\alpha)_{2n}=2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$$

Proof:- $(\alpha)_{2n}=(\alpha)(\alpha+1)(\alpha+2)(\alpha+3)......(\alpha+2n-1)$

$=[\ (\alpha)(\alpha+2)......(\alpha+2n-2)\ ][\ (\alpha+1)(\alpha+3).......(\alpha+2n-1)\ ]$

$=2^{2n}[\ (\ \frac{\alpha}{2}\ )(\ \frac{\alpha+2}{2}\ )(\ \frac{\alpha+4}{2}\ )..........(\ \frac{\alpha+2n-2}{2}\ )\ ]$

$[\ (\ \frac{\alpha+1}{2}\ )(\ \frac{\alpha+1}{2}+\ )...........(\ \frac{\alpha+1}{2}+n-1\ )\ ]$

$=2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$

Similarly, we can show that

$(\alpha)_{kn}=K^{kn}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)..........\left(\frac{\alpha+k-1}{k}\right)_n$

Example 3: show that

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

Proof:

$$\Gamma(\alpha+n) = (\alpha+n-1)(\alpha+n-2)\ldots\ldots\ldots\alpha\Gamma(\alpha)$$
$$= (\alpha)(\alpha+1)\ldots\ldots\ldots(\alpha+n-1)\Gamma(\alpha)$$
$$\Gamma(\alpha+n) = (\alpha)_n\Gamma(\alpha)$$
$$\therefore (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

4.0 ACTIVITY I

Express $\int_0^{\frac{\pi}{2}} \sin\alpha\ \theta d\theta$ in terms of Beta function.

5.0 CONCLUSION

In this unit, we have studied Gamma function, Beta function and Factorial notations. You are required to study these functions because you would be required to apply them in future.

6.0 SUMMARY

The study of special functions in mathematics is of significant importance. Study this area properly before moving to the next unit.

7.0 TUTOR-MARKED ASSIGNMENT

i. The Beta function of p, and, q is defined by the integral

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, (p,q>0).$$

By writing $t = \sin^2\theta$ obtain the equivalent form

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta d\theta, (p,q>0)$$

ii. Show that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

iii. By writing $t = x/(x+a)$ in the definition of Beta function, show that

$$\int_0^\infty \frac{x^{p-1}dx}{(x+a)^{p+q}} = a^{-q}B(p,q)$$

8.0 REFERENCES/FURTHER READING

Earl, A. Coddington (1989). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Francis, B. Hildebrand (2014). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.

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UNIT 2 HYPER GEOMETRIC FUNCTION

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider a class of function usually referred to as hyper-geometric functions. The series solution of the associated differential equation usually takes the form of a geometric series. Most often, hyper-geometric equation has $x = 0, x = 1$ and $x = \infty$ as regular points and ordinary point elsewhere.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine the differential equations that can give rise to hyper-geometric functions;
- explain the properties of this functions; and
- apply this function where necessary.

3.0 MAIN CONTENT

3.1 Hyper-Geometric Function

The solutions of the differential equation $x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - aby = 0$ (1)
are generally called Hyper-geometric functions.

Note that $a, b,$ and c are fixed parameters.

We solve this equation (1) about the regular singular point $\underline{x = 0}$
Shifting the index

$$\sum_{n=0}^{\infty} n(n+c-1)e_n x^{n-1} - \sum_{n=1}^{\infty} (n+a-1)(n+b-1)e_{n-1}x^{n-1} = 0$$

For $n \geq 1$

$$e_n = \frac{(n+a-1)(n+b-1)}{n(n+c-1)} e_{n-1}$$

$$e_n = \frac{a(a+1)(a+2)...(a+n-1).b(b+1)(b+2)...(b+n-1)e_0}{n!.c(c+1)(c+2)(c+2).....(c+n-1)}$$

Using factorial notation, we have

$$e_n = \frac{(a)_n (b)_n}{n!.(c)_n} e_0$$

Let us choose $e_0 = 1$

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

We have the symbol

$2F_1(a,b,c,x)$ to represent solution

$$y_1 = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

$$2F_1(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^{n.}$$

The solution is valid in $0 < |x| < 1$. The other root of the indicial equation is (1-c). We may put $y = \sum_{n=1}^{\infty} f_n{}^n x^{n+1-c}$

For the moment, let c be not an integer for (1), the indicial equation has root zero and i-c. Let $y = \sum_{n=1}^{\infty} e_n x^{n+r}$

$$\sum_{n=1}^{\infty} e_n x^{n+b}(n+b-1)x^{n+b} - \sum_{n=0}^{\infty} e_n (n+b)(n+b-1)x^{n+b} + c\sum_{n=0}^{\infty} e_n (n+b)(n+b-1)x^{n+b} + c\sum_{n=0}^{\infty} e_n (n+b)x^{n+b-1}$$

$$-(a+b+1)\sum_{n=0}^{\infty} e_n x^{n+b} = 0$$

or

$$\sum_{n=0}^{\infty} e_n (n+b)(n+b-1+c)x^{n+b-1}$$

$$\sum_{n=0}^{\infty} e_n \left[ab+1)(a+b+1)(n+b)(n+b)(n+b-1) \right] x^{n+b} = 0$$

The indicial equation is

$$e_n (b)(b-1+c) = 0$$

(Note c is not an integer).

Corresponding to $b = 0$,

$$\sum_{n=1}^{\infty} n(n+c-1)e_n = \sum_{n=0}^{\infty} (n+a)(n+b)e_n x^n = 0$$

Example 1: If $R(c-a-b)>0$ and if c is neither zero nor a negative integer,

$$2F_1(a,b,c,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

Proof

$$2F_1(a,b,c,1)$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

Example 2: Show that

(a) $2F\ (\alpha,\beta,\beta,x)=(1-x)^{-\alpha}$

(b) $x2F(1;1;2;-x)=Log(1+x)$

Solution:

(a) $2F\ (\alpha,\beta,\beta,x)$

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}x^n$$

$$=1+\alpha x+\frac{\alpha(\alpha+1)}{21}x^2+......\frac{\alpha(\alpha+1)(\alpha+1)}{31}x^3+......=(1-x)^{-\alpha}$$

(b) $x2F(1;1;2;-x)=Log(1+x)$

$$x[\ 1+\frac{1.1}{1.2}(-x)+\frac{1.2.1.2}{1.2.2.3}(-x)^2+\frac{1.2.3.1.2.3}{1.2.3.2.34}(-x)^3$$

$$y_2=x^{1-c}2F_1(a+1-c,b+1-c;2-c;x)$$

Example 3: If $z\le t\le 1$, and if $R(c)>R(b)>0$,

$$2F_1(a;b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^a\,dt$$

Proof

Beta-function now

$$\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}=\int_0^1 t^{b-1}(1-t)^{c-b-1}dt$$

Also

$$\frac{(b)_n}{(c)_n}=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{(b+n)(\Gamma(c-b)}{\Gamma(c+n)}$$

Thus

$$2F_1(a;b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(a)_n (zt)^n dt}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^a dt$$

Where

$$(1-z)^a = \sum_{n=0}^{\infty} \frac{(-a)(-a-1)...(-a-n+1)(-1)^n y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{a(a+1)...(a+n-1)y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{a(a+1)...(a+n-1)y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n y^n}{n!}$$

4.0 ACTIVITY II

Find a general solution in terms of hyper geometric functions $8x(1-x)y'' + (4-14x)y' - y = 0$

5.0 CONCLUSION

You have learnt in this unit some properties of hyper-geometric functions. You are requested to study this unit properly before going to the next unit.

6.0 SUMMARY

Recall that you learnt about the class of differential equation, which usually give rise to hyper-geometric functions. You also learnt about the relations of this function to Gamma and Beta functions. Study this unit properly before going to the next unit.

7.0 TUTOR-MARKED ASSIGNMENT

i. If R(c – a – b) >0 and if c is neither zero nor a negative integer show that

$$2F_1(a;b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

ii. Show that

(a) $2F_1(\partial;\beta;\beta;x) = (1-x)^{-\alpha}$

(b) $x^2 F_1(1;1;2;-x) = \log(1+x)$

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UNIT 3 BESSEL FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In solving differential equation, we often come across some problems which exhibit some characteristic which needed to be studied further. Such equations are Legendry equation and Bessel equations. We shall study in detail in this unit the Bessel equation which gives rise to Bessel functions. This is because of the wide applicability of this function in physics and applied mathematics.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify Bessel functions correctly; and
- solve problems related to Bessel functions.

3.0 MAIN CONTENT

3.1 Bessel Function

The equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-v^2)y=0 \qquad (1)$$

is called Bessel's equation of index $v.$

(i) x = 0 is the regular Singular point of the equation (1) in the finite plane
(ii) Assume that $v.$ is not integer.

$$y=\sum_{n=0}^{\infty} c^m x^{m+r}$$

Substituting this expression and its first and second derivatives into Bessel equation, we obtain

$$=\sum_{n=0}^{\infty}(m+z)(m+z-1)c_m x^{m+r}+\sum_{m=0}^{\infty}(m+r)c_m x^{m+r}+\sum_{m=0}^{\infty}c_m x^{m+r+2}-v^2\sum_{m=0}^{\infty}c_m x^{m+z}=0$$

(a) $r(r-1)c_0+rc_0-v^2c_0=0 \qquad (m=0)$

(b) $(r+1)(r)c_1+(r+1)c_1-v^2c_1=0 \qquad (m=1)$

(c) $(m+r)(m+r-1)c_m+c_{m-2}-vc_m=0 \qquad (m=2,3,....$

Now $c_0\neq 0,$ thus the indicial equation from (a) $(r+v)(r-v)=0$

The roots are $r=v, r=-v$

$r_1-r_2=2v$

$v\neq 0,$

$v\neq$ integer

$2v$ is integral multiply of v, i.e. v is zero or $+ve$ integer

Now we obtain the solution corresponding to the value $r=v.$

From (b) we obtain $c_1=0$ (c) can be written

$(m+r-v)(m+r+v)c_m+c_{m-2}=0$

Since $c_1=0$, it follows that $c_3=c_5=c_7=.......0.$ Thus we can replace m by $2m.$

$(2m+r-v)(2m+r+v)c_m+c_{2m-2}=0$

Now $r=v$

$(2m+2v)(2m)c_{2m}+c_{2m-2}=0$

$\therefore c_{2m}=-\dfrac{c_{2m-2}}{2^2(v+m)m}$ (but v is not an integer) $\qquad (m=1,2,....)$

Assume

$$c_0=\frac{1}{2^v\Gamma(v+1)}$$

$$c_2=\frac{c_0}{2^2(v+1)}=\frac{-1}{2^{v+2}1!\Gamma(v+2)}$$

$$c_4=\frac{c_2}{2.2^2(v+2)}=\frac{1}{2^{v+4}2!\Gamma(v+3)}$$

$$c_{2m}=\frac{(-1)^m}{2^{2m+v}m!\Gamma(v+m+1)}$$

Thus, the solution is

$$y=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+v}m!\Gamma(v+m+1)}$$

We denote this solution by the notation

$$J_v(x)=x^v\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+v}\Gamma(m+v+1)m!} \qquad (2)$$

$J_v(x)$ is called the <u>Bessel Function of the first kind</u> of order v.

By Ratio test we know that the series converges for all values of x
Replacing v by $-v$, we have

$$J_{-v}(x) = x^{-v}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-v}\Gamma(m-v+1)m!} \qquad (3)$$

(2) and (3) are the independent solutions.

Thus $y = c_1 J_v(x) + c_2 J_{-v}(x)$

(i) If $v = 0$, then the solution $J_v(x)$ and $J_{-v}(x)$ are identical. One can verify from (2) and (3)

(ii) If v is $+ve$ integer, then the second solution $J_{-v}(x)$ is not independent of $J_v(x)$

Say $v = n$ then the factor

$$\frac{1}{\Gamma(m-n+1)} = \frac{1}{(m-n)!} \text{ in (3) is zero} \qquad (4)$$

When $m < n$ hence (3) is equivalent to

$$J_{-n}(x) = x^{-n}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n}(m-n)!m!} \qquad (5)$$

Replace m by $m+n$ in (5), we get change the index

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+n}}{m!(m+n)!} \qquad (6)$$

From (2), when $v = n$ integer, thus

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\left(\frac{x}{2}\right)^{2m+n}}{m!(m+n)!} \qquad (7)$$

From (6) and (7), we get

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (8)$$

Further properties of Bessel functions of first kind
From (2)

$$x^v J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2v}}{2^{2m+v}\Gamma(m+v+1)m!}$$

Now we use the formula

$$\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$$

$$\frac{d}{dx}[x^v J_n(x)] = \sum_{m=0}^{\infty} \frac{2(-1)^m x^{2m+2v-1}}{2^{2m+v}\Gamma(m+v)m!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2v-1}}{2^{2m+v-1}m!\Gamma(m+v)}$$

$$= x^v J_{v-1}(x)$$

Thus we obtain

$$\frac{d}{dx}\left[\ x^{v} J_{v}(x)\ \right] = x^{v} J_{v-1}(x) \qquad (9)$$

Similarly, we can show that

$$\frac{d}{dx}\left[\ x^{-v} J_{v}(x)\ \right] = (-1) x^{-v} J_{v+1}(x) \qquad (10)$$

(9) can also use written

$$vx^{v-1} J_{v}(x) + x^{v} J_{v}{}'(x) = x^{v} J_{v-1}(x) \qquad (11)$$

(10) Can also be written

$$-vx^{-v-1} J_{v}(x) + x^{-v} J_{v}{}'(x) = -x^{-v} J_{v+1}(x) \qquad (12)$$

Multiplying (12) by x^{2v} and subtracting from (11), we have

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_{v}(x) \qquad (13)$$

Multiplying (12) by x^{2v} and adding with (11), we get

$$J_{v-1}(x) - J_{v+1}(x) = 2J_{v}'(x) \qquad (14)$$

(i) we know that

$$Sinx = x - \frac{x^3}{31} + \frac{x^5}{51} - \frac{x^7}{7} + \ldots\ldots\ldots.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Now $(2k+1)! = \Gamma(2k+2) = (2)_{2k}$

$$(2)_{2k} = 2^{2k}(1)_k (3/2)_k$$

$$= 2^{2k} k!\left(\ \frac{3}{2}\ \right)k$$

$$\frac{2^{2k} k!\Gamma\left(\ k+1+\frac{1}{2}\ \right)}{\Gamma\left(\ \frac{3}{2}\ \right)}$$

But $\Gamma\left(\ \frac{3}{2}\ \right) = \frac{1}{2}\Gamma\left(\ \frac{1}{2}\ \right) = \frac{1}{2}\sqrt{\pi}$

$$\therefore (2k+1)! = \frac{2^{2k+1} k!\Gamma\left(\ k+1+\frac{1}{2}\ \right)}{\sqrt{\pi}}$$

$$\therefore Sinx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}\sqrt{\pi}}{2^{2k+1} k!\Gamma(k+1+1/2)} \qquad (15)$$

If we take $V = \frac{1}{2}$, then from (2), we have

$$J_{\frac{1}{2}}(x) = x^{\frac{1}{2}} \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+\frac{1}{2}} k! \Gamma(k + \frac{3}{2})} \qquad (16)$$

From (15) and (16), we have

$$J_{\frac{1}{2}}(x) = \sqrt{\tfrac{2}{\pi x}} \sin x$$

In similar manner, by considering the expansion

$\cos z = \sum_{n=0}^{\infty} \frac{(-1) z^{2n}}{(2n)!}$, we obtain

The formula

$$J_{-\frac{1}{2}}(x) = \sqrt{\tfrac{2}{\pi x}} \cos x$$

Examples: Show that:

(i) $J_o(x) = -1 J_1(x)$

(ii) $J_n'' = \frac{1}{4}(J_{n-2} - 2J_n + J_n + 2)$

(iii) $J_1(x) = J_0(x) - \frac{1}{x} J_1(x)$

(iv) $J_2(x) = (1 - \frac{4}{x^2}) J_1(x) - \frac{2}{x} J_0(x)$

(v) $\int x^m J_n(x) dx = x^m J_{n+1}(x) - (m - n - 1) \int x^{m-1} J_{n+1} dx(x)$

Solution

$$\int x^m J_n(x) dx = x^{m-n-1} [x^{n+1} J_n(x)] dx$$

$$\int x^{m-n-1} \frac{d}{dx} [x^{n+1} J_{n+1}(x)] dx$$

Integrating by parts, we have

$$= x^m J_{n+1}(x) - (m - n - 1) \int x^{m-1} J_{n+1}(x) dx$$

This proves the result

Prove that:

(vi) $\int J_n(x) dx = -x^m - J_{n-1} + (m + n - 1) \int x^{m-1} J_{n-1}(x) dx$

(vii) $\int J_{v+1}(x) dx = \int J_{v-1}(x) dx - 2J_v(x)$

It immediately follows from the identity $2J_v'(x) = J_{v-1}(x) - J_{v+1}(x)$

(viii) $\int J_{v-1}(x) dx = -x^{1-v} J_{v-1}(x) + c$

(ix) $\int x^{1-v} J_{v1}(x) dx = -x^{1-v} J_{v-1}(x) + c$

(x) $\int x^3 J_0(x)dx = -x^3 J_1(x) - 2^2 xJ_2(x) + c$

Problem: Defining the Bessel function $J_n(x)$ by means of the generating function

Example (a) $\exp\{\frac{1}{2}x(t-t^{-1})\} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

We have LHS

$$e^{\frac{x}{2}t} \cdot e^{-\frac{x}{2}\frac{1}{t}} = \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^m}{m!} t^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!} t^{-k}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{\substack{m-k=n \\ m,k\geq 0}}^{\infty} \frac{(-1)^k (\frac{x}{2})^{m+k}}{m!k!} \right) t^n$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+n}}{(n+k)!k!} \right) t^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x) t^n = \text{RHS}$$

show that,

If n is an integer

(b) $J_{-n}(x) = (-1)^n J_n(x)$

(c) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

(d) $J_{n-1}(x) + J_{n+1}(x) = 2J'_n(x)$

Solution

(b) Replace t by $-\frac{1}{t}$ in the definition

$$\exp\{\frac{1}{2}x(t-\frac{1}{t}) = \sum_{n=\infty}^{\infty} (-1)^n t^{-n} J_n(x)$$

$$= \sum_{n=\infty}^{\infty} (-1)^{-n} t^n J_{-n}(x)$$

$$= \sum_{n=\infty}^{\infty} t^n J_n(x)$$

Thus we get

$J_{-n}(x) = (-1)^n J_n(x)$

Example (b): Prove that

$$j_o(z) = \frac{1}{2}\int_0^{2\pi} \cos(z\cos\theta)d\theta$$

Proof: We know that

$$\exp[\frac{z}{2}(t - t^{-1}) = \sum_{n\to\infty}^{\infty} j_n(z)t^n$$

Put
$t = e^{i\theta}$, take real parts of both sides and integrate between 0 and 2 ………..

$$\exp[\frac{z}{2}(e^{i\theta} - e^{-i\theta})] = e^{iz\sin\theta} = \cos(z\sin\theta) + i\sin(z\sin\theta) = \sum_{n=\infty}^{\infty} j_n(z)(\cos n\theta + i\sin n\theta)$$

$$Cos(z\sin\theta) + i\sin(z\sin\theta) = j_0(z) + \sum_{n=1}^{\infty} j_n(z)\cos n\theta + i\sum_{n=0}^{\infty} J_n(z)\sin(z\sin n\theta)$$

$$\int_0^{2\pi} Cos(z\sin\theta)d\theta = \int_0^{2\pi} j_0(z)d\theta + \sum_{n=1}^{\infty} j_n(z)\int_0^{2\pi}\cos n\theta d\theta$$

$$\int_0^{2\pi} Cos(z\sin\theta)d\theta = 2\pi J_0(z) + 0$$

Hence

$$j_o(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\cos\theta)d\theta$$

Example (c): Prove that

$$\int_0^{\frac{\pi}{2}} J_0(z\cos\theta)\cos\theta d\theta = \frac{\sin z}{z}$$

Solution

$$j_p(z) = (\tfrac{z}{2})^p \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m+p)!}(\frac{z}{2})^{2m}$$

Put $p = 0$

$$j_0(z) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!m!}(\frac{z}{2})^{2m}$$

Replace z by $z\cos\theta$ multiply both sides of integrate between 0 to $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} j_0(z\cos\theta)\cos\theta d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-1)z^{2m}}{m!m!2^{2m}}\int_0^{\frac{\pi}{2}} j_0(z\cos\theta)^{2m} d\theta$$

Now $\int_0^{\frac{\pi}{2}} (\cos\theta)d\theta = \frac{2^{2m}(m!)^2}{(2m+1)!} \quad = \sum_{m=1}^{\infty} \frac{(-1)^m z^{2m}}{(2m+1)1}$

Now we know that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}.............$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!}.............$$

$$= \frac{\sin z}{z}$$

3.1.1 Bessel Functions of the First Kind

In the definition of Bessel function $j_p(z)$ put $z = iy$, then (p integer)

$$j_p(iy) = e^{i\frac{p\pi}{2}} j_p(y)$$

$$j_p(iy) = e^{i\frac{p\pi}{2}} (\frac{y}{2})^p \sum_{m=0}^{\infty} \frac{1}{m!(m+p)!} (\frac{y}{2})^{2m}$$

(2) Bessel function of the second kind

Solution

$$\int_0^z tJ_n(at)J_n(bt)dt$$

$$\frac{z\{aJ_n(bZ)J_n{}'(az) - bJ_n(az)J_n'(bz)\}}{b^2 - a^2}$$

Solution

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2)y = 0$$

$$y_1 = J_n(at), y_2 = J_n(bt)$$

(i) $t^2 y_1'' + ty_1' + (a^2t^2 - n^2)y_1 = 0$

(ii) $t^2 y_2'' + ty_1' + (b^2t^2 - n^2)y_2 = 0$

Multiply (i) by y_2 and (ii) by y_1, and subtracts, we find

$$t(y_2y_1'' - y_1y_2'') + (y_2y_1' - y_1y_2') = (b^2 - a^2)ty_1y_2$$

Or

$$\frac{d}{dt}[t(y_2 y_1' - y_1 y_2')] = (b^2 - a^2) t y_1 y_2$$

Integrating with respect to t from o to z yield

$$(b^2 - a^2)\int_0^z t(y_1 y_2)dt = t(y_2 y_1' - y_1 y_2')$$

4.0 ACTIVITY III

prove that $\int_0^{\frac{\pi}{2}} J_0\,(z\cos\theta)\cos\theta\, d\theta = \frac{\sin z}{z}$

5.0 CONCLUSION

We have considered Bessel function in its general setting in this unit. You are required to read this unit carefully before going to the next unit.

6.0 SUMMARY

Recall that Bessel functions are usually associated with a class of equations called Bessel equations. They are usually denoted by the notation:

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} r(m+v+1)m1}$$

We gave some examples to enable you understand the contents of this unit. We also examined another type of Bessel function usually referred to as Bessel Function of the First Kind. However, you are to master this unit properly before going into the next unit.

7.0 TUTOR-MARKED ASSIGNMENT

i. Given that

$$e^{\frac{x}{2}(r-\frac{1}{r})} = \sum_{n=-\infty}^{\infty} r^n J_n(x)$$

Deduce that $(n+1)J_{n+1}(x) = \frac{x}{2}[J_n(x) + J_{n+2}(x)]$

ii. Obtain the general solution of each of the following equations in terms of Bessel functions, or if possible in terms of elementary functions.

(a) $x\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + xy = 0$ (b) $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 0$ (c) $x^4\frac{d^2y}{dx^2} + a^2y = 0$

8.0 REFERENCES/FURTHER READING

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