MODULE 3 SPECIAL FUNCTIONS AND PARTIAL DIFFERENTIAL EQUATION

UNIT 1 LEGENDRY FUNCTION

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider another class of special functions which has wide application in physical problems. This class of functions has orthogonality properties. The functions are legendry functions.

2.0 OBJECTIVES

At the end this unit, you should able to:

- identify legendry functions and legendry polynomial;
- solve problems relating to legendry functions; and
- determine the properties of legendry functions and legendry polynomial.

3.0 MAIN CONTENT

3.1 Legendry Functions

The Legend differential equation of order *n* is given by:

$$(1-x^2)\frac{d^2y}{dx_2}-2x\frac{dy}{dx}+p(p+1)y=0$$

The solution of this equation is known as Legendry function

$$(1-x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} c_n n x^{n-1} + p(p-1)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} \{(n+2)(n+1)c_{n+2} - c_n[n(n+1) - p(p+1)]\}x^n = 0$$

Recurrence relation

$$(n+2)(n+1)c_{n+2} = c_n(n^2 + n - p^2 - p) \qquad (2)$$

Thus

$$c_{n+2} = \frac{(p-n)(p+n+1)}{(n+2)(n+1)}c_n$$

Therefore

$$c_2 = \frac{p(p+1)}{2!}c_o$$

$$c_3 = \frac{(p-1)(p+2)}{3!}c_1$$

$$c_4 = \frac{(p-2)(P+3)}{4\cdot 3}c_2 \qquad \frac{(p-2)(p)(p+1)(P+3)}{4!}c_0$$

$$c_5 = \frac{(p-3)(P+4)}{5.4}c_3$$

$$\frac{(p-3)(p-1)(p+2)(P+4)}{5!}c_1 \text{ etc.}$$

$$y_1 = 1 - p(p+1)\frac{x^2}{2!} + (p-2)p(p+1)(p+3)\frac{x^4}{4!} - - - - - - - -$$

$$y_2 = x - (p-1)(p+2)\frac{x^3}{3!} + (p-3)(p-1)(p+2)(p+4)\frac{x^5}{5!} - - - - - - - -$$

$$P_n(x) = \sum_{k=0}^{m} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}$$

Where m is the largest integer and runs greater than $\frac{n}{2}$.

In particular

$$p_o(x) = 1 \qquad p_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Example 1: show that:

$p_1(x) = x$

$\int_{-1}^{1} p_m(z) p_n(z) dz = 0$ if $m \neq n$

Solution:

$$(1-z^2)[p_n p_m'' - p_m p_n''] - 2z(p_n p_m' - p_m p_n')$$
$$= [n(n+1) - m(m+1)] P_m P_n$$

and subtracting, we have

$$(1-z^2)\frac{d}{dz}[p_n p_m{}' - p_m p_n{}'] - 2z[p_n p_m' - p_m p_n']$$
$$= [n(n+1) - m(m+1) p_n p_m$$
$$\frac{d}{dz}[(1-z^2)(p_n p_m{}' - p_m p_n')] = [n(n+1) - m(m+1)] p_n p_m$$

Integrate from -1 to 1 we have

$$[n(n+1) - m(m+1)] \int_0^1 p_m(z) p_n(z) dz$$
$$(1-z^2)(p_n p_m' - p_n p_m')1_{-1}^{1} = 0$$

Example 2: show that:

ii $\int_{-}^{1} p_m(z) p_n(z) dz = \frac{2}{2n+1}$, if $m = n$

Solution:

$$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} p_n(z) t^n$$

Square is

$$\frac{1}{1-2zt+t^2} = \left(\sum_{n=0}^{\infty} p_n(z) t^n \right)^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_m(z) p_n(z) t^{m+n}$$

Integrating from -1 to 1

$$\int_{-1}^{1} \frac{dz}{1-2zt+t^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{ \int_{-1}^{1} p_m(z) p_n(z) dz \} t^{m+n}$$

$$\sum_{n=0}^{\infty} \{ \int_{-1}^{+1} [p_n(z)]^2 dz \} t^{2n}$$

L.H.S $= -\frac{1}{2t} \log(1-2zt+t^2)1_{-1}^{1} = \frac{1}{t} \text{ in } \frac{1+t}{t-1}$

$$=\sum_{n=0}^{\infty}\{\frac{2}{2n+1}\}t^{2n}$$

Equating the coefficients, we have the result

Example 3: show that:

(iii) $(n+1)p_{n+1}(z)-(2n+1)zp_n(z)+np_{n-1}(z)=0$

Solution: Differentiating with respect to t both sides of the identity $\frac{1}{\sqrt{1-2zt+t^2}}=\sum_{n=0}^{\infty} p_n(z)t^n$ and

Multiply by $1-2zt+t^2$, we have

$$(z-t)\sum_{n=0}^{\infty} p_n(z)t^n=(1-2z+t^2)\sum_{n=0}^{\infty} np_n(z)t^{n-1}$$

Or

$$\sum_{n=0}^{\infty} zp_n(z)t^n-\sum_{n=0}^{\infty}p_n(z)t^{n+1}=\sum_{n=0}^{\infty}np_n(z)t^{n-1}-\sum_{n=0}^{\infty}2nzp_n(z)t^n+\sum_{n=0}^{\infty}np_n(z)t^{n+1}$$

Equating the coefficient problem: Show that

$$(1-2xz+z^2)^{-½}=p_0(x)+p_1(x)z+p_2(x)z^2+..........=\sum_{n=0}^{\infty}p_n(x)z^n$$

Proof:-

$$(1-2xz+z^2)^{\frac{1}{2}}=1+\frac{1}{2}(2xz-z^2)+$$

$$\frac{\frac{1}{2}\frac{3}{2}(2xz-z^2)^2}{2!}+\cdots\cdots+\frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})....[\frac{(2p-1)}{2}]}{P!}(2xz-z^2)^p$$

The power of z^p can only occur in the term going from the pth term $(2xz-z^2)^p$ $[=z^p(2x-z)$ down. Thus, expanding the various powers of $(2x-z)$, we find that the Coefficient of z^p is

$$\frac{(\frac{1}{2})(\frac{3}{2})......[\frac{(2p-1)}{2}]}{p!}(2x)^p$$

Prove that

$$p_n(z)=\frac{1}{2^n n1}\frac{d^n}{dz^n}(z^2-1)^n$$

$$p_n(z) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)}{2^n r1(n-r)1(n-2r)1} z^{n-2r}$$

Where p is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$.

$$= \frac{1}{2^n n!} \frac{d^n}{dz^n} \sum_{r=0}^{\infty} \frac{(-1)^2 n!}{r!(n-r)!} z^{2n-2r}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dz^n} \sum_{r=0}^{\infty} \frac{(-1)^2 n!}{r!(n-r)!} z^{2n-2r}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n$$

3.1.1 Legendry Polynomial

The equation $(1-x^2)\frac{d^2 y}{dx} - 2x\frac{dy}{dx} + n(n+1)y = 0$
is called Legendry equation.

(i) $x = +1$ are the regular singular points of the equation. We solve the equation with the singular point $x = 1$, we put $x = 1-u$ and obtain the transformed equation.

(ii) $u(u+2)\frac{d^2 y}{du^2} + n - (n+1)y = 0$ is the regular singular point.

We assume the solution point

$$y = \sum_{k=0}^{\infty} a_k u^{k+c}$$

$$\frac{dy}{du} = \sum_{k=0}^{\infty} a_k u^{k+c-1}$$

$$\frac{d^2 y}{du^2} = \sum_{k=0}^{\infty} (k+c)(k+c-1)a_k u^{k+c-2}$$

The roots of the indicial equations are $c = 0{,}0\,0$. Hence one solution is logarithmic. We are only interested here in the non-logarithmic solution.
Hence

$$y = \sum_{k=0}^{\infty} a_k u^k$$

We assume a_0 is non-zero arbitrary constant, and

$$a_k = \frac{-(k-n-1)(k+n)}{2k^2} a_{k-1}$$

Solving the recurrence solution, we have

$$a_k = \frac{(-1)^k(-n)_k(1+n)_k a_0}{2^k(k!)^2}$$

Thus the solution is

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k(-n)_k(n+1)_k(1-x)^k}{2^k(k!)^2}$$

Where a_0 = 1

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{(-1)_k(n+1)_k}{(1)_k(k1)}\left(\frac{1-x}{2}\right)^k$$

$$y_1 = 2F_1[-n, n+1; 1; (\frac{1-x}{2})]$$

$$= P_n(x).$$

$P_n(x)$ is called the Legendry Polynomials

It is customary to take

$$c_n = \frac{2n!}{2^n(n!)^2},\ n = 0,1,2,...$$

But from (3)

$$c_{n-2} = -\frac{n(n-1)}{(2n)(2n-1)}c_n, \qquad \text{or}$$

$$c_n = -\frac{(2n)(2n-1)}{n(n-1)}c_{n-2}, \qquad c_n = -\frac{(2n-2)!}{2^n(n-1)!(n-2)}$$

$$c_{n-4} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)}$$

or

$$c_{n-2k} = \frac{(2n-2k)!(-1)^k}{2^n k!(n-k)(n-2k)!}$$

Then the legendry Polynomials of degree n is given by

$$P_n(x) = \sum_{K=0}^{M} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

integer not greater that $\frac{n}{2}$.

(1) $P_n(x) = \sum_{K=0}^{M} \frac{(-1)^k}{2^n k!(n-k)!} \frac{d^{n-1}}{dx^{n-1}} (x^{2n-2k-1})$

Since

$$\frac{d}{dx^n}(x^{2n-2k}) = (2n-2k)\frac{d}{dx^{n-1}}(^{2n-2k-1})$$
$$(2n-2k)(2n-2k-1).......(n-2k+1)x^{n-2k}$$
$$\frac{(2n-2k)!}{(n-2k)!}x^{n-2k}$$

Hence

$$P_n(x) = \sum_{K=0}^{M} \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{K=0}^{M} \frac{(-1)^k n!(x^2)^{n-k}}{k!(n-k)!}$$

We may now extend the range of this sum by taking k range from 0 to *n*. This extension will not affect the result, since the added terms are a polynomial of degree less than n and the nth derivative will vanish.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{K=0}^{M} \frac{(-1)^n n!(x^2)^{n-k}}{k!(n-k)!}$$

and by binomial theories, we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^{n,} \qquad n = 0,1,2,..........$$

This is known as Rodrigues formula

Example: Show that

$$P_2(x) = \frac{1}{2}(3x^{2,} - 1)$$

Solution: By Rodrigues' formula

$$P_2(x) = \frac{1}{4\cdot 2!}\frac{d^2}{dx^2}(x^{2,}-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4 - 2x^2 + 1)$$
$$= \quad \frac{1}{2}(3x^2 - 1)$$

Examples: Show that

(i) $P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x).\ n = 1,2,........ (1)$

(ii) $P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x).$ (2)

Solution

(i) $$P'_{n+1}(x) = \frac{d}{dx}[\frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1}]$$

$$= \frac{1}{2(n+1)2^n n!}\frac{d^n}{dx^n}[\frac{d^2}{dx^2}(x^2-1)^{n+1}]$$

$$= \frac{1}{2(n+1)2^n n!}\frac{d^n}{dx^n}[4n(n+1)x^2(x^2-1)^{n-1} + 2(n+1)(x^2-1)^n]$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}[2n(x^2-1+1)(x^2-1)^{n-1} + (x^2-1)^n]$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}[(2n+1)(x^2-1)^n + 2n(x^2-1)^{n-1}]$$

$$P_{n+1}{}'(x) = [(2n+1)P_n(x) + P'_{n-1}(x)$$

Solution (ii)

Now we have that

$$\frac{d}{dx}[xf(x)] = f(x) + xf'(x)$$
$$\frac{d^2}{dx^2}[xf(x)] = xf''(x) + 2f'(x)$$

and in general

$$\frac{d^{p+1}}{dx^{p+1}}[xf(x)] = \frac{d^{p+1}}{dx^{p+1}}[x(x^2-1)^n]$$

Now

$$p'_{n+1}(x) = \frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}[x(x^2-1)^n]$$
$$\frac{1}{2^n n!}[x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n + (n+1)\frac{d^n}{dx^n}(x^2-1)^n]$$
$$= xp'_{n+1}(x) + (n+1)p_n(x)$$

4.0 CONCLUSION

You have learnt about legendry polynomial and legendry functions in this unit. Read this unit properly before going to the next unit.

5.0 SUMMARY

You will recall that the legendry polynomial is defined as:

$$P_n(x) = \sum_{K=0}^{M} \frac{(-1)^k (2n-2k)!}{2^n k! n-k! n-k! n-2k!} x^{n-2k}$$

This polynomial has Orthogonality property which we have mentioned in this unit.

6.0 TUTOR-MARKED ASSIGNMENT

i. Show that the substitution $t = 1 - x$ transform Legendre's equation to the form:

$$t(2-t)\frac{d^2y}{dt^2} + 2(1-t)\frac{dy}{dt} + p(p+1)y = 0$$

ii. Problem: Show that

a. $P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x).\ n = 1,2,........$

b. $P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x).$

7.0 REFERENCES/FURTHER READING

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UNIT 2 SOME EXAMPLES OF PARTIAL DIFFERENTIAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

A partial differential equation is an equation that contains one or more partial derivatives. Such equations occur frequently in application of mathematics. We shall only discuss certain partial differential equations which are used frequently in applied mathematics. In fact, we are going to discuss a kind of boundary value problems which enters modern applied mathematics at every turn.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- recognise partial differential equations by type and character;
- explain the methods of solving partial differential equations; and
- apply the knowledge in some other related field.

3.0 MAIN CONTENT

3.1 Some Examples of Partial Differential Equations in Applied Mathematics

Many linear problems in applied mathematics involve the solution of an equation obtained by specialising the form.

$$\Delta^2\theta + f = \lambda \frac{d^2\theta}{dt^2} + \mu \frac{d\theta}{dt} \qquad (1)$$

Where *f* is a specified function of position and λ and μ are certain specified physical constant. Here, Δ^2 is the Laplacian operator in one, two or dimension under consideration and is of the form.

$$\Delta^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \qquad (2)$$

In rectangular co-coordinator of three space, the unknown function ϕ is the function of the position co-ordinates (x, y, z) and the tine t.

(i) Laplace Equation

$$\Delta^2\theta = 0 \qquad (3)$$

It is satisfied by the velocity potential in an ideal incompressible fluid without vertical or continuously distributed sources; and by gravitational potential in free space; electrostatic potential in the steady flow of electric currents in solid conductors, and by the steady-state temperature distribution in solids.

(ii) Poisson's Equation

$$\Delta^2\theta + f = 0 \qquad (4)$$

is satisfied, for example, by the velocity potential of an incompressible, irrotational, ideal fluid with continuously distributed sources or by steady temperature distribution due to distributed heat sources, and by a 'sheds function' involved in the elastic torsion of prismatic bars, with a suitably prescribed function f.

(iii) Wave Equation

$$\Delta^2\theta = \frac{1}{c}\frac{\partial^2\theta}{\partial t^2} \qquad (5)$$

This arises in the study of propagation of waves with velocity *c*, independent of the wave length. In particular, it is satisfied by the components of the electric or magnetic vector in electromagnetic theory, by suitably chosen component of displacement, in the theory of elastic vibrations, and by the velocity potential in the theory of sound (acoustics) for a perfect gas.

(iv) The Equation of Heat Conduction

$$\Delta^2\theta = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t} \qquad (6)$$

This is satisfied, for example, by the temperature at a point of a homogeneous body and by the concentration of a diffused substance in the theory of diffusion, with a suitably prescribed constant θ.

(v) The Telegraphic Equation

$$\frac{\partial^2\theta}{\partial x^2}+\lambda\frac{\partial\theta}{\partial t}+\mu\frac{\partial\theta}{\partial t}=0 \qquad (7)$$

This is one dimensional specialisation of (1), and is satisfied by the potential in a telegraph cable, where $\lambda = Lc$ and $\mu = Rc,$ if the Leakage is neglected (L is inductance, c capacity and R resistance per unit length).

(vi) Differential equation of higher order, involving the operator Δ^2, are rather frequently encountered, in particular, the bi-Laplacian equation in two dimensions.

$$\Delta^4\theta=\Delta^2\Delta^2\theta=\frac{\partial^u\theta}{\partial x^u}+2\frac{\partial^2\theta}{\partial x^2\partial y^2}+\frac{\partial\theta}{\partial y^u}=0 \qquad (8)$$

is involved in many two dimensional problem of the theory of elasticity.

The solution of a given problem must satisfy the proper differential equation, together with similarly prescribed boundary condition or initial conditions (2 f time is involved).

The above equation can be changed to cylindrical co-ordinates $r\theta z,$ related to x, y and z by the equations

$$x = rCos\theta, y = rSin\theta, z = z$$

$$\Delta^2\theta=\frac{\partial^2\theta}{\partial r^2}+\frac{1}{2}\frac{\partial\theta}{\partial r}+\frac{1}{2}\frac{\partial\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\theta}{\partial\theta^2}+\frac{\partial^2\theta}{\partial z^2}=0 \qquad (9)$$

In spherical co-ordinates P,θ,θ related to x,y,z by the equations $x = PSin\theta Cos\theta, y = PSin\theta, z = PCos\theta$. Laplace' equation is

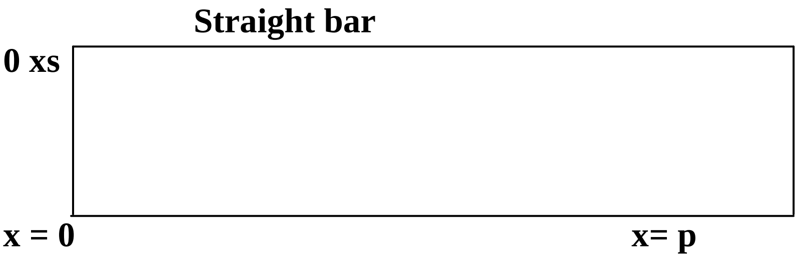
$$\frac{\partial^2 v}{dp^2}+\frac{2}{p}\frac{\partial v}{\partial\theta}+\frac{\partial^2 v}{\partial\theta^2}+\frac{\cot\theta}{p^2}\frac{\partial v}{\partial\theta}+\frac{Cos^2\theta}{p^2}\frac{\partial^2 v}{\partial\theta^2}=0. \qquad (10)$$

In what now follows we shall solution methods of partial differential equations:

Method of separation of variables
Consider the equation

$$\alpha^2\frac{\partial^2 v}{dx^2}=\frac{\partial u}{dt}, 0 < x < l, t > 0 \qquad (a)$$

This is called the heat conduction equation



This is a straight bar of uniform cross section and homogenous material. The temperature v can be considered constant on any given cross section.

$v = U(x,t).$

α^2 is a constant known as $v = U(x,t).$ In addition, we shall assume that the ends $v = U(x,t).$ of the bar are held at temperature zero: thus $v = 0$ when and $x = 0.$

$\therefore u(o,t) = 0, \quad u(l,t) = 0, \qquad t > 0,$ (1)

Finally, the initial distribution of temperature in the bar is assumed to be given thus $U(x,o) = h(x) \qquad 0 \le x \le l$ (2)

(1) and (2) are called boundary conditions

We assume that

$u(x,t) = f(x)g(t)$ (3)

Substituting equation (3) for $u(x,t)$ in (a) yields

$\alpha^2 f''(x)g(t) = f(x)g'(t)$ (4)

or

$\alpha^2 \frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)}$ (5)

Now equation (5) is said to have its variable separated; that is, the left member of equation (5) is a function of x alone and the right member of equation (5) is a function of x alone.

Since x and t are independent variables, the only way in which a function of x alone can equal to function of t alone is for each function to be constant.

$\therefore \frac{f''(x)}{f(x)} = b$ (6)

$\alpha^2 \frac{g'(x)}{g(x)} = b$ (7)

In which b is arbitrary

The partial differential equation (a) has now been replaced by two ordinary differential equations. This is the essence of the **method of separation of variables.**

Boundary Conditions

$xv(o,t) = f(o)g(t) = 0$ (8)

by (1), if $g(t)$= 0 then $u(x,t)$ will be identically zero. It is not acceptable because it does not satisfy the equation (2). Thus it must satisfy the condition

$f(o) = 0$ (9)

Similarly, the boundary condition at $x(l)$ $U(l,t) = 0$ requires
$f(l) = 0$ (10)

There are two possible values of the constant k i.e. $k = 0$ or $k \neq 0$.
Values of the constant k :

(i) $k = 0$, then the general solution of equation (6) is

$f(x) = c_1 x + c_2$ (11)

(9) Must satisfy the boundary value conditions (9) and (10). In order to satisfy (9)
$f(o) = c_1 \cdot 0 + c_2 = 0 \Rightarrow c_2 = 0$ (12)

It is also satisfies the equation (10)
$\therefore f(l) = c_1 l = 0 \Rightarrow c_1 = 0$ Since $l \neq o$.
$\therefore c_1 = 0$ (13)

Hence, $f(x)$ is identically zero, and therefore $U(x,t)$ is also identical zero
(ii) $k \neq 0$, we take $k = -\lambda^2$, where λ is a new parameter. Thus, the equation (6) becomes
$f''(x) + \lambda^2 f(x) = 0$ (14)

and its general solution is
$f(x) = b_1 e^{i\lambda x} + b_2 e^{i\lambda x}$ (15)

Applying the boundary condition (9) and (10), we have

$$\left.\begin{array}{l} b_1 + b_2 = 0 \\ b_1 e^{i\lambda l} + b_2 e^{i\lambda l} = 0 \end{array}\right\} \quad (16)$$

The system (16) has a trivial solution $b_1 = 0$ and $b_2 = 0$ always, but it is not acceptable $u(x,t)$ is identically zero. Non-trivial solution exists if and only the determinant.

$$\begin{vmatrix} 1 & 1 \\ e^{i\lambda l} & e^{-i\lambda l} \end{vmatrix} = 0 \qquad (17)$$

If we write, $\lambda = \mu + iv$ then

$e^{-\mu il} e^{vl} - e^{i\mu l} e^{-vl} = 0$ or

$e^{vl}(\cos \mu l - i \sin \mu l) - e^{-vl}(\cos \mu l + i \sin \mu l) = 0$

$$\left.\begin{aligned} \cos \mu l(e^{vl} - e^{-vl}) = 0 \\ \sin \mu l(e^{vl} + e^{-vl}) = 0 \end{aligned}\right\} \qquad (18)$$

Now $\cos \mu l(e^{ve} - e^{-ve}) > 0$ for v and l, thus $\sin \mu l = 0 \Rightarrow v \neq 0$ (19) must be so chosen that

$$\mu = \frac{n\pi}{l}, \qquad (20)$$

where n is a non-zero integer. From (16) $b_1 = -b_2$ (21)

From (15), we have

$$f(x) = b_1(e^{i\lambda l} - e^{-i\lambda l}) = \frac{b_1}{2}\left(\frac{e^{i\lambda l} - e^{-i\lambda l}}{2}\right)$$

Thus, $f(x)$ is proportional to $\sin \lambda lx$ (23)

$$k = -\lambda^2 = -\frac{n^2\pi^2}{l^2} \qquad (24)$$

Where n is an integer

From (7), we have (25)

Hence the function

$$u_n(x,t) = c_n \exp[-\frac{n^2\pi^2\alpha^2 t}{l^2}]\sin\frac{n\pi x}{l} \qquad (26)$$

$n = 1,2,3,.......$ where c_n is an arbitrary constant, satisfies the boundary conditions $2,9,10$ as well as the differential equation (a). The functions u_n are sometimes called fundamental solution of the heat conduction problem (a) (1) and (2).

By the boundary condition (2) we get from (26).

$$u_n(x,o) = c_n \sin\frac{n\pi x}{l} \qquad (27)$$

For $n = 1,2,...........$

Each solution given by (27) satisfies the differential equation and the boundary condition. Since partial differential equation involved is linear and homogeneous in u and its derivatives, a sum of solution are also a solution. From the known solutions, $u_{1,}u_2.........u_n,..........$ we may thus construct others with sufficiently strong convergence condition. It is true that even the infinite series

$$u = \sum_{n=1}^{\infty} u_n \text{ or}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp[-\frac{n^2\pi^2\alpha^2 t}{l^2}]\sin\frac{n\pi x}{l} \qquad (28)$$

is a solution of the differential equation. In order to satisfy the initial condition (2) we must have

$$u(x,o) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = h(x) \qquad (29)$$

Now let us suppose that it is possible to express $h(x)$ by means of an infinite series forms

$$h(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (30)$$

We know how to compute b_n i.e

We can satisfy the equation (29) by choosing $c_n = b_n$ for each n. With the coefficient selected in this manner, equation (28) gives the solution of the boundary value problem (a) (1) and (2)

Thus, we have solved the problem consisting of the heat condition equation.

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < l, \qquad t > o \qquad (1)$$

The boundary condition

$$u(o,t) = 0, \qquad u(l,t) = 0, \qquad t > o \qquad (2)$$

and the initial condition

$$u(x,o) = h(x), \qquad 0 \le x \le l \qquad (3)$$

we found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp[\frac{n\pi^2\alpha^2 t}{l^2}]\sin\frac{n\pi x}{l} \qquad (4)$$

With the coefficients b_n are the same as in the series

$$h(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (5)$$

Where

$$b_n = \frac{2}{l}\int_0^l h(x)\sin\frac{n\pi x}{l}\,dx \qquad (6)$$

The series in equation (5) is just the Fourier

Example 2: If we consider the problem of the heat conduction equation of boundary conditions and the initial condition, the boundary conditions are known as non-homogeneous boundary condition.

Solution: If we shall reduce the present problem to one having homogeneous boundary condition, we use the physical argument. After a long time, i.e. …………., we anticipate a steady state temperature distribution ……………….. will be reached, and must satisfy difficulties (1), then (which is independent of time t and initial condition).

…………. (4)

and it satisfied the boundary condition

…………. (5)

Which apply even as …………….. The solution (4) with condition (5)

………….. (6)

Hence the steady state temperature is a linear function of x.

We shall express $U(x,t)$ as the sum of the steady state temperature and another distribution w(x, t).

$$\therefore U(x,t) = U(x) + w(x,t) \qquad (7)$$

(7) satisfies (1), we have

$$\alpha^2 (u+w)_{xx} = (u+w)_t \qquad (8)$$

It follows that

$$\alpha^2 W_{xx} = W_t \qquad (9)$$

Now boundary condition

$$w(o,t) = u(o,t) - u(o) = T_1 - T_1 = 0 \qquad (10)$$

$$w(l,t) = u(l,t) - u(l) = T_2 - T_2 = 0 \qquad (11)$$

The initial condition

$$w(x,o) = (u(x,o) - u(x) = f(x) - u(x) \qquad (12)$$

Where u(x) is given by (6)
The problem now becomes precisely the previous one and we have the solution

$$W(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-\frac{n^2\pi^2\alpha^2 t}{l} \right] Sin\frac{n\pi x}{l} \qquad (13)$$

Where

$$b_n = \frac{2}{l}\int_0^l W(x,0) Sin\frac{n\pi x}{l} dx \qquad (14)$$

Where

$$U(x,t) = (T_2 - T_1)\frac{x}{l} + T_1 + \sum_{n=1}^{\infty} b_n \exp\left[-\frac{n^2\pi^2\alpha^2 t}{l} \right] Sin\frac{n\pi x}{l} \qquad (15)$$

Where

$$b_n = \frac{2}{l}\int_0^l \left[f(x) - (T_2 - T_1)\frac{x}{l} - T_1 \right] Sin\frac{n\pi x}{l} dx \qquad (16)$$

Example 3: Now we consider the problem of the heat conduction equation with the boundary condition

$$\alpha^2 U_{xx} = U_t \qquad (1)$$

$$U_x(0,t) = 0 \ , \quad U_x(l,t) = 0, \qquad t > 0 \qquad (2)$$

and the initial condition

$$U(x,0) = f(x) \qquad (3)$$

Solution: We solve the equation by the method of separation of variables. [When the ends of the bar are insulated so that there is no passage of heat through them].

$$U(x,t) = h(x)g(t) \qquad (4)$$

(4) satisfies (1), we have

$$\frac{h''(x)}{h(x)} = \frac{1}{\alpha^2}\frac{g'(t)}{g(t)} = \sigma \quad (5)$$

We assume that σ is real, we consider three cases σ = 0 and –ve.

(i) If α = 0 then equation (5) given

$$U(x,t) = K_1 x + K_2 \quad (6)$$

Applying boundary condition (2), we get

$K_1 = 0.$ Hence corresponding

$h''(x) = 0$

$h(x) = C_1 x + C_2$

$$g'(t) = 0 \to g(t) = C_B \quad (7)$$

Solution is 0

(ii) If $\sigma = \lambda^2$ where λ is real and +ve

$$\therefore h''(x) - \lambda^2 h(x) = 0 \quad (8)$$

$$g'(t) - \lambda^2 \alpha^2 g(t) = 0 \quad (9)$$

From (8) and (9), we have

$$U(x,t) = e^{\alpha^2 \lambda^2 t}(k_1 Sinh\lambda x + k_2 Cosh\lambda x). \quad (10)$$

Now we apply the boundary condition (2)

$U_x(x,t) = e^{\alpha^2 \lambda^2 t}(k_1 \lambda Cosh\lambda x + K_2 \lambda Sinh\lambda x)$

$U_x(o,t) = 0$ and $U_x(l,t) = 0$

$K_1 \lambda = 0 \Rightarrow K_1 = 0, l \neq 0$

$K_2 \lambda Sin\lambda l = 0 \Rightarrow K_2 = 0, l \neq 0$

$\to K_1 = 0$ and $K_2 = 0$

This is not acceptable, because it does not satisfy the initial condition of some examples of partial differential equation, hence σ, can not be positive.

(iii) $\sigma = -\lambda^2$, where λ is +ve and real. From (8) and (9), we obtain

$$U_x(x,t) = e^{\alpha^2 \lambda^2 t}(k_1 \lambda Cos\lambda x - K_2 \lambda Sin\lambda x) \quad (11)$$

Now we apply boundary condition, we get $(K_1 = 0)$ $K_1 = 0$ and $x = l$ $\lambda = \frac{n\pi}{l}$ for n=1,2,…… $(Sin\theta = 0, \theta = n\pi)$

$\therefore \sigma = -(\frac{n^2\pi^2}{l^2})$, where n is +ve integer (12)

Combining the solution, we have

$$U_o(x,t) = \frac{1}{2}\text{Co} \quad (13)$$

$$U_n(x,t) = C_n \exp[\ \frac{n^2\pi^2\alpha^2 t}{l^2}\ ] Cos\frac{n\pi x}{l}, n = 1, 2, ... \quad (14)$$

These solution functions satisfy the differential equation (1) and bounding conditions (2) for any value of the constant C_n. Both differential equation and boundary values are linear and homogeneous, any finite sum of the fundamental solutions will also satisfy them. We will assume that this is also line for convergent infinite sums of fundamental solution as well.

Thus

$$U(x,t) = \frac{1}{2}Co + \sum_{n=1}^{\infty} c_n \exp[\ \frac{n^2\pi\ \alpha^2 t}{l^2}\ ] Cos\frac{n\pi x}{l} \quad (15)$$

Where C_n are determined by the initial requirement that

$$U(x,o) = \frac{1}{2}Co + \sum_{n=1}^{\infty} c_n Cos\frac{n\pi x}{l} = f(x) \quad (16)$$

Thus, the unknown coefficients in equation (15) must be coefficients in the Fourier Cosine series of period $2l$ for f. Hence

$$c_n = \frac{2}{l}\int_o^l f(x) Cos\frac{n\pi x}{l} dx, n = 0, 1, 2, \ldots$$

With this choice of the coefficients, (15) provides the solution of the equation.

Example: Elastic string with non-zero initial displacement

First, suppose that string is displaced from its equilibrium position, and then released with zero velocity at time t = 0 to vibrate freely. Then in vertical displacement U(x,t) must satisfy the wave equation.

$$\alpha^2 U_{xx} = U_{tt}, \qquad 0 < x < l, t > 0 \quad (1)$$

The boundary conditions are

$$U(o,t) = 0, \quad U(l,t) = 0, \quad t \geq 0 \quad (2)$$

and the initial conditions

$U(x,o) = f(x), U_t(x,0) = 0, \quad 0 \le x \le l$ (3)

Where f is given function describing the configuration of the string at $t = 0$.

Solution: We use the equation (1) by the method of separation of variables.

Assuming that

$U(x,t) = X(x) \quad T(t)$ (4)

Substituting u in (1), we get

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T''}{T} = \sigma \quad (4)$$

We assume that σ is real (we shall prove it somewhere else. We consider these cases σ = 0, -ve and +ve.

(i) $\sigma = 0$, then $X'' = 0$, and $X(x) = K_1 x + K_2$ (5)

(ii) If $\sigma > 0$, then $X'' - \lambda^2 x = 0$ and $X(x) = K_1 Sinh\lambda x + K_2 Cosh\lambda x$ (6)

Where $\lambda = \sqrt{\sigma}$

Consider the solution given by (5)

$U(x,t) = (K_1 x + K_2)T(t)$

By boundary conditions $U(o,t) = 0$

$U(o,t) = (0 + K_2)T(t) = 0$, $T(t)$ can not be zero, because U(x,t) will be identically zero. Thus, $K_2 = 0$. Next consider the second boundary condition $U(l,t) = 0$ then $U(l,t) = (K_1 l)T(t) = 0 \Rightarrow K_1 = 0$

Thus X(x) = 0, it is not acceptable.

(iii) Similarly, we can show that for (6) under boundary condition $K_1 = 0 = K_2$.

Thus, $\sigma = 0$ and $\sigma = +ve$ real number are not acceptable. We now consider the last cast

(ii) $\sigma = -\lambda^2 \ne 0$ (7)

$X'' + \lambda^2 X = 0$ (8)

$T'' + \lambda^2 \alpha^2 T = 0$

$\therefore X(x) = K_1 Sin\lambda x + K_2 Cos\lambda x$ (9)

$T(t) = K_3 Sin\lambda\alpha t + K_4 Cos\lambda\alpha t$ (10)

Thus

$$U(x,t) = (K_1 Sin\lambda x + K_2 Cos\lambda x)(K_3 Sin\lambda\alpha t + K_4 Cos\lambda\alpha t) \quad (11)$$

Satisfies (1) for all values of K_1, K_2, K_3, K_4 and for $\lambda > 0.$

Now we impose the boundary conditions

$U(o,t) = 0,$ Thus

$$U(o,t) = K_2(K_3 Sin\lambda\alpha t + K_4 Cos\lambda\alpha t) = 0 \Rightarrow K_2 = 0 \quad (12)$$

Secondly, boundary condition $U(l,t) = 0,$

$$U(l,t) = (K_1 Sin\lambda l)(K_3 Sin\lambda\alpha t + K_4 Cos\lambda\alpha t) = 0 \quad (13)$$

If $K_1 = 0,$ then $U(x,t)$ U is zero identically, thus for non-trivial solution.

$$Sin\lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}, n = 1,2........ \quad (14)$$

Hence the functions which satisfy the equation (1) and boundary condition (2) are of the form.

$$U_n(x,t) = Sin\frac{n\pi x}{l}(C_n Sin\frac{n\pi\alpha t}{l} + K_n Cos\frac{n\pi\alpha t}{l}) \quad (15)$$

Where n=1,2,…….. C_n and K_n are arbitrary constants. Now we apply the principle of super position of solution and assume that

$$U_n(x,t) = \sum_{n=1}^{\infty} U_n(r,t) = \sum_{n=1}^{\infty} Sin\frac{n\pi x}{l}(C_n Sin\frac{n\pi\alpha t}{l} + K_n Cos\frac{n\pi\alpha t}{l}) \quad (16)$$

Further, we assume that (16) can be differentiated term by term with respect to t $U_t(x,o) = 0$ yields

$$U_n(x,o) = \sum_{n=1}^{\infty} C_n Cos\frac{n\pi\alpha t}{l} Sin\frac{n\pi x}{l} = 0 \quad (17)$$

$\Rightarrow C_n = 0$ for all values of n

$\Rightarrow$ The other condition $U(x,o) = f(x)$

Given

$$U_n(x,o) = \sum_{n=1}^{\infty} K_n Sin\frac{n\pi x}{l} = f(x) \quad (18)$$

Consequently, K_n must be line coefficients in the Fourier Series of period $2l$ for f and are given by

$$K_n = \frac{2}{l}\int_0^l f(x) Sin\frac{n\pi x}{l}dx, n = 1, 2, ... \qquad (19)$$

Thus, the formal solution of the problem (1) with condition (2) and (3) is

$$U_n(x,t) = \sum_{n=1}^{\infty} K_n Sin\frac{n\pi x}{l} Cos\frac{n\pi\alpha t}{l} \qquad (20)$$

Where the coefficients K_n are given by (19).

For a fixed value of n the function

$Sin\frac{n\pi x}{l} Cos\frac{n\pi\alpha t}{l}$ is periodic in time t with the period $\frac{2l}{n\alpha}$; it therefore represents a vibratory motion of the string having this period or having the frequency $\frac{n\pi\alpha t}{l}$. The quantities $\lambda a = \frac{n\pi a}{l}$ for n=1, 2,…… are the natural frequencies of the string. The factor $K_n Sin\frac{n\pi x}{l}$ represents the displacement pattern occurring in the string, when it is executing vibrations of the given frequency.

In the case of heat conduction problem, it is attempting to try to show this by directly substituting equation (20) for $U(x,t)$ in (1), (2) and (3), we compute.

$$U_{xx}(x,t) = -\sum_{n=1}^{\infty} K_n (\frac{n\pi}{l})^2 Sin\frac{n\pi x}{l} Cos\frac{n\pi a t}{l} \qquad (21)$$

Due to the presence of the factor n^2 in the numerator, this series may not converge. It is not possible to justify directly with respect to either variable in $(0,l)$ and $t > 0$, provided h is twice continuously differentials on $(-\infty,\infty)$. This require f' and f'' are continuous on $[0,l]$. Furthermore, since h'', we must have

$f''(o) = f''(l) = 0$

Example: General problem for inelastic string.

Consider the equation

$$\alpha^2 U_{xx} - U_{tt} = 0, \quad 0 < x < l, t > 0; \qquad (1)$$

The boundary condition

$$U(0,t) = 0, \quad U(l,t) = 0 \qquad (2)$$

and the initial conditions

$$U(x,o) = f(x), U_t(x,o) = g(x), 0 \le x \le l \qquad (3)$$

Solution: As we have done in the previous case, we obtain the solution

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} Sin\frac{n\pi x}{l}\left(C_n Sin\frac{n\pi\alpha t}{l} + K_n Cos\frac{n\pi\alpha t}{l}\right) \qquad (4)$$

Applying the initial condition $U(x,0) = f(x)$ yields

$$U(x,o) = -\sum_{n=1}^{\infty} K_n Sin\frac{n\pi x}{l} = f(x) \qquad (5)$$

Where the coefficients K_n are given in the Fourier Sine Series of period 2l for f and are given

$$K_n = \frac{2}{l}\int_0^l f(x) Sin\frac{n\pi a}{l}dx, n = 1, 2, ... \qquad (6)$$

Differentiate (4) with respect to t and putting substitution. We establish the validity in a different way. We show

$$U(x,t) = \frac{1}{2}[\ h(x-\alpha t) + h(x+\alpha t)\ ] \qquad (7)$$

Where h is function obtained by extending the initial data $f(x)$ into $(-l, 0)$ as an odd function, and other values of x as a periodic function on period $2l$.

That is

$$h(x+2l) = h(x).$$

Now

$$h(x) = \sum_{n=1}^{\infty} K_n Sin\frac{n\pi x}{l}$$

Then

$$h(x-\alpha t) = \sum_{n=1}^{\infty} K_n (Sin\frac{n\pi x}{l} Cos\frac{n\pi\alpha t}{l} - Cos\frac{n\pi x}{l} Sin\frac{n\pi\alpha t}{l})$$

$$h(x+\alpha t) = \sum_{n=1}^{\infty} K_n (Sin\frac{n\pi x}{l} Cos\frac{n\pi\alpha t}{l} + Cos\frac{n\pi x}{l} Sin\frac{n\pi\alpha t}{l})$$

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} K_n Sin\frac{n\pi x}{l} Cos\frac{n\pi\alpha t}{l}$$

Equation h is the function (20) follows on adding the two equations.

Note: [If $f(x)$ has a Fourier series, then it must be periodic and continuous].

1. If $U(x,t)$ is continuous for $0 < x < l$ and $t > 0$ provided that h is requires that f is continuous on line interval $(-\infty,\infty)$. This requires that f is continuous on the interval (o,l) and, since h is odd periodic extension of f, that f be zero at $x = 0$ and $x = p$.
2. U is twice continuously differentiable $t = 0$, we get

$$U_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi\alpha}{l} C_n Sin \frac{n\pi\alpha x}{l} = g(x) \qquad (7)$$

Hence the coefficients $(\frac{n\pi a}{l})C_n$ are the coefficients in the Fourier Sine series of period 2l for g: Thus

$$(\frac{n\pi a}{l})C_n = \frac{2}{l}\int_0^l g(x) Sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \ldots \qquad (8)$$

Thus, the equation (4) with the equation (6) and (8) constitutes the formal solution of the equation (1).

Example: Laplace equation: One of the most important of all the partial differential equations occurring in applied mathematics is associated with the name of Laplace. Here is Laplace equation in two dimensions.

$$U_{xx} + U_{yy} = 0 \qquad (a)$$

and in 3 dimension

$$U_{xx} + U_{yy} + U_{zz} = 0 \qquad (b)$$

Now solve (1) under the boundary condition. The problem of finding a solution of Laplace equation which takes on given boundary values is known as *Dirichilet* problem.

Problem I: Solve the Laplace equation

$$U_{xx} + U_{yy} = 0 \qquad (1)$$

In the rectangle $0 < x < a, \quad 0 < y < b,$ and which satisfies the boundary condition

$$U(x,0) = 0, \qquad U(x,b) = 0, \qquad 0 < x < a, \qquad (2)$$
$$U(o,y) = 0, \qquad U_{xx}(a,y) = f(y), \qquad 0 \le y \le b$$

Where f is given function on $0 \le y \le b$

Solution:

$U(x, y) = X(x)Y(y)$ (3)

Substituting $U(x, y)$ in (1), we get

$\frac{X''}{X} = -\frac{Y''}{Y} = K$ (4)

We assume that k is real.

(i) If K=0 then $X'' = 0$ and $Y'' = 0$, and $U(x, y) = (K_1 x + K_2)(C_1 y + C_2)$ (5)

The homogeneous boundary conditions $y = 0$ and $y = b$ can be satisfied by $C_1 = C_2 = 0 \Rightarrow U(x, y),$ is identically zero. Hence $k = 0$ is not acceptable.

(ii) $K = \lambda^2, \lambda > 0,$ then

$X'' - \lambda^2 X = 0$

$Y'' + \lambda^2 Y = 0$

and thus

$U(x, y) = (K_1 Sinh\lambda x + K_2 Cos\lambda x)(C_1 Sin\lambda y + C_2 Cos\lambda y)$ (6)

In order to satisfy the boundary conditions $x = 0$ and $y = 0 \Rightarrow K_2 = 0 = C_2$

The condition at $y = b$ becomes

$K_1 C_1 Sin\lambda x Sin\lambda b = 0$ (7)

$\Rightarrow Sin\lambda b = 0$

It follows that

$\lambda b = n\pi, n = 1, 2, 3, \ldots$ (8)

Thus, the solution of the differential equation must be of the forms.

$U_n(x, y) = C_n Sinh\frac{n\pi x}{b} Sin\frac{n\pi y}{b}, n = 1, 2, 3, \ldots,$ (9)

There functions are the fundamental solution of the present problems. We assume

$U_{xx}(x, y) = \sum_{n=1}^{\infty} U_n(x, y) = \sum_{n=1}^{\infty} C_n \frac{n^2\pi^2}{b^2} Sinh\frac{n\pi x}{b} Sin\frac{n\pi y}{b}$ (10)

Now the last boundary conditions

$U_{xx}(a, y) = f(y) = \sum_{n=1}^{\infty} C_n \frac{n^2\pi^2}{b^2} Sinh\frac{n\pi a}{b} Sin\frac{n\pi y}{b}$ (11)

Thus, the coefficients $C_n Sinh\frac{n\pi a}{b}$ must be the coefficients in the Fourier Sine series of period $2b$ for $f(y)$ and are given by

$$C_n Sinh\frac{n\pi a}{b} = \frac{2}{b}\int_o^b f(y) Sin\frac{n\pi y}{b}dy \qquad (12)$$

Thus, (10) is the solution of the equation (1) satisfying the boundary condition (2) and coefficients $C_n Sinh\frac{n\pi a}{b}$ are computed from (12).

(iv) If $K = -\lambda^2$ then

$$X'' + \lambda^2 X = 0$$
$$Y'' - \lambda^2 Y = 0$$

and

$$U(x,y) = (K_1 Sin\lambda x + K_2 Cos\lambda x)(C_1 Sinh\lambda y + C_2 Cosh\lambda y) \qquad (13)$$

Again, the boundary condition at $y = 0$ and $y = b$ lead to $C_1 = C_2 = 0$, so again $U(x,y)$ is zero, everywhere. Hence $K = -\lambda^2$ is not acceptable.

Problem: Dirichilet problem for a circle

Consider the Laplace equation in polar co-ordinates

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0 \qquad (1)$$

With boundary condition

$$U(0,\theta) = f(\theta) \qquad (2)$$

f is a given function on $0 \le \theta \le 2\pi.$

Moreover, in order that $U(r,\theta)$ the single valued, it is necessary that, as a function of θ, U must be periodic with period $2\pi.$

Solution: Let $U(r,\theta) = R(r)\theta(\theta)$ (3)

We substitute (3) in (1)

$$R''\theta + \frac{1}{r}R'\theta + \frac{1}{r^2}R\theta'' = 0$$

Or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\theta''}{\theta} = K \qquad (4)$$

Again, we assume that the separation constant must be real.

(i) Suppose K=0

$$r^2 R'' + rR'\theta = 0$$
$$\theta'' = 0$$
$$\therefore U(r) = K_1 + K_2 Log\, r$$

$$\theta(\theta) = C_1 + C_2\theta$$

$$\therefore U(r,\theta) = (K_1 + K_2 Log\, r)(C_1 + C_2\theta) \quad (5)$$

Since equation is periodic in θ, thus $C_2 = 0$.

Further $r \to 0$ the term log r is unbounded. This behaviour is unacceptable. Thus, we impose the condition that $U(r,\theta)$ remains finite at all points of the circle and hence we must take $K_2 = 0$

$$\therefore U_0(r,\theta) = \text{Constant} = \frac{1}{2}c_0 \text{ say} \quad (6)$$

(ii) If $K = -\lambda^2$ then

$$\theta'' - \lambda^2\theta = 0 \quad (7)$$

$$\therefore \theta(\theta) = C_1 e^{\lambda\theta} + C_2 e^{-\lambda\theta} \quad (8)$$

The function $U(r,\theta)$ is periodic thus $C_1 = C_2 = 0$.
This makes $U(r,\theta)$ identically zero. This is not acceptable.

(iii) Finally, $K = \lambda^2, \lambda > 0,$ yields

$$r^2R'' + rR'' - \lambda^2 R = 0 \quad (9)$$

And

$$\theta'' + \lambda^2\theta = 0 \quad (10)$$

$$\therefore R(r) = K_1 r^{\lambda} + K_2 r^{-\lambda}$$

$$\theta(\theta) = C_1 Sin\lambda\theta + C_2 Cos\lambda\theta \quad (11)$$

In order that for θ to be periodic with period 2π, it is necessary that λ be a positive integer.

Moreover, the solution $r^{-\lambda}$ of (10) be discarded, since it becomes unbounded as $r \to o$. Consequently, $K_2 = 0$. Hence the solutions (1) are

$$U_n(r,\theta) = r^n(C_n Cos(n\theta) + K_n Sin(n\theta)), n = 1, 2, \ldots,$$

These functions, together with that of equations (6), serve as fundamental solutions of the present problem. Thus

$$U(r,\theta) = \frac{1}{2}Co + \sum_{n=1}^{\infty} r^n(C_n Cos(n\theta) + K_n Sin(n\theta)) \quad (13)$$

The boundary condition (2) then requires that

$$f(\theta) = U(a,\theta) = \frac{1}{2}Co + \sum_{n=1}^{\infty} a^n(C_n Cos(n\theta) + K_n Sin(n\theta)) \quad (14)$$

for $0 \le \theta \le 2\pi.$

The function $f(\theta)$ may be extended outside the interval. So also it is periodic of period $2\pi,$ and has a Fourier series of the function (14).

$$a^n C_n = \frac{1}{2}\int_0^{2\pi} f(\theta) Cos(n\theta) d\theta \qquad (15)$$

$$a^n K_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) Sin(n\theta) d\theta \qquad (16)$$

With this choice of coefficients (13) represents the solutions of the boundary value problem of equations (1) and (2).

(i) The heat conduction equation in two space dimension may be expressed in terms of polar co-ordinates as

$$\alpha^2 (U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}) = U_t$$

Assuming that

$U(r,\theta,t) = R(r)\theta(\theta)T(t).$ Find ordinary equation satisfied by $R(r), \theta(\theta),$ and $T(t)$.

$I_1 = \int_{-C}^{C} Cos\frac{n\pi x}{C} Sin\frac{k\pi x}{C} dx = 0$ for all k and n.

The integrand is an even function.

$$I_2 = \int_0^C (1 - Cos\frac{2n\pi x}{C}) dx$$

$$= [\ x - \frac{C}{2n\pi} Sin\frac{2n\pi x}{C}\ ]_0^c = C. n = 1, 2, \ldots$$

Similarly, we can obtain

$$I_3 = \int_{-c}^{C} Cos^2 \frac{n\pi x}{c} dx = C \quad \text{for} \quad n = 1, 2, 3, \ldots$$

$$= 2C \text{ for } n = 0.$$

Similarly, we can show that

$$I_4 = \int_{-c}^{C} Sin\frac{n\pi x}{c} Sin\frac{k\pi x}{c} dx = C \qquad \text{if } k = n$$

$$I_5 = \int_{-c}^{C} Cos\frac{n\pi x}{c} Cos\frac{k\pi x}{c} dx = C \qquad \text{if } k = n$$

Periodic Function: A function f is said to be periodic with period T if the domain of f contains $x + T$ whenever it contains x, and y.

$$f(x+T) = f(x) \text{ for every value of } x.$$

$f(x+T) = f(x)$

With fundamental period $T = 2l/m$, every such function has the period $2l$.

The function $Sin\frac{m\pi x}{l}$ and $Cos\frac{m\pi x}{l}$, for $n = 1, 2, ...$ are periodic.

3. Fourier Series

We assume that there exists a series expansion of the type

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n Cos\frac{n\pi x}{C} + b_n Sin\frac{n\pi x}{C}] \qquad (1)$$

Valid in the interval $-C \le x \le C$

$$j_p(z) = (\frac{z}{2})^p \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m+p)!}(\frac{z}{2})^{2m}$$

(1) is called the Fourier series corresponding to $f(x), a_n$ and b_n.

Multiply (1) by $Sin(\frac{K\pi x}{C})dx$, where k is a +ve integer, and then integrate each term from –c to c, thus arriving at

$$\int_{-c}^{C} f(x)Sin\frac{k\pi x}{c}dx = \frac{1}{2}a_0\int_{-c}^{c} Sin\frac{k\pi x}{C}dx + \sum_{n=1}^{\infty} a_n \int_{-c}^{c} Cos\frac{n\pi x}{C}Sin\frac{k\pi x}{C}dx + \sum_{n=1}^{\infty} b_n \int_{-c}^{c} Sin\frac{n\pi x}{C}Sin\frac{k\pi x}{C}dx \qquad (2)$$

As seen earlier

$$\int_{-c}^{C} Cos\frac{n\pi x}{C}Sin\frac{k\pi x}{C}dx = 0 \text{ for all } k \text{ and } n.$$

And

$$\int_{-c}^{C} Sin\frac{n\pi x}{c}Sin\frac{k\pi x}{C}dx = 0 \text{ if } k \ne n$$

$$= c \quad \text{if } k = n$$

Using (2), we have

$$\int_{-c}^{C} f(x)Sin\frac{k\pi x}{C}dx = cb_k, \quad k = 1, 2, 3, \ldots$$

or

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)Sin\frac{n\pi x}{C}dx, \quad n = 1, 2, 3, \ldots$$

Let us now evaluate the coefficients $a_{n.}$ using the multiplies $Cos\frac{k\pi x}{C}dx$ throughout equation (1) and then integrating term by term for –c to c, we get

$$\int_{-c}^{C} f(x)Cos\frac{k\pi x}{C}dx = \frac{1}{2}a_o\int_{-c}^{c} Cos\frac{k\pi x}{C}dx + \sum_{n=1}^{\infty} a_n \int_{-c}^{c} Cos\frac{n\pi x}{C}Cos\frac{k\pi x}{C}dx + \sum_{n=1}^{\infty} b_n \int_{-c}^{c} Sin\frac{n\pi x}{c}Cos\frac{k\pi x}{c}dx \qquad (4)$$

Now we know that

$$\int_{-c}^{C} Cos\frac{n\pi x}{C}Cos\frac{k\pi x}{C}dx = 0 \quad \text{for } n \ne k$$

$$= c \quad \text{for } n = k$$

If $k \neq 0$, (4) reduces to

$$\int_{-c}^{C} f(x) Cos \frac{k\pi x}{C} dx = ca_k$$

or

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x) Cos \frac{n\pi x}{c} dx \quad (5)$$

Next we determine the coefficient a_0. Suppose $K = 0$ in (4)

$$\int_{-c}^{C} f(x)dx = \frac{1}{2}a_o\int_{-c}^{c} dx + \sum_{n=1}^{\infty}[a_n\int_{-c}^{c} Cos \frac{n\pi x}{C} dx + b_n\int_{-c}^{c} Sin \frac{n\pi x}{c} dx] \quad (6)$$

Thus we have

$$\int_{-c}^{C} f(x)dx = \frac{1}{2}a_0(2c)$$

or $a_0 = \frac{1}{c}\int_{-c}^{C} f(x)dx$ (7)

Thus we write the formal expansion as follows

$$f(x) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}[a_n Cos \frac{n\pi x}{C} + b_n Sin \frac{n\pi x}{c} dx] \quad (8)$$

With

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x) Cos \frac{n\pi x}{C} dx, n = 0,1,2\ldots, \quad (9)$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x) Sin \frac{n\pi x}{C} dx, n = 1,2,\ldots, \; \ldots\ldots. \quad (10)$$

Note that the formulae (9) and (10) depend only upon the values of $f(x)$ in the interval $-c \leq x \leq c.$ Since each of the terms in the Fourier series (8) is periodic with period $2c$, the series converges for all x whenever it converges in $-c \leq x \leq c.$, and its sum is also a periodic function with period 2c. Hence $f(x)$ is determined for all x by its values in the interval $-c \leq x \leq c.$.

4.0 ACTIVITY II

Find the Laplace transform of $\sin \mu t$ given $L(\sin t) = \frac{1}{s^2 + 1}$

5.0 CONCLUSION

You have been introduced to partial differential equation in this unit. The attempts here are just introductory. You are required to study this unit properly because you will refer to it in your subsequent courses in mathematics.

6.0 SUMMARY

In this unit, various forms and types of partial differential equations were studied. These include (1) Wave equation (2) Laplace equation and (3) Heat equation. We also proposed various methods of solving these equations which include method of separation of variables and Fourier series applications. You are required to study this unit properly and attempt all the exercises at the end of the unit.

7.0 TUTOR-MARKED ASSIGNMENT

i. Show that the boundary-value problem
$\frac{d^2y}{dx^2} - k^2 y = 0, y(0) = y(l) = 0$ cannot have a nontrivial solution for real values of k

ii. Determine those values of k for which the partial differential equation
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ possesses nontrivial solutions of the form $T(x, y) = f(x)\sinh ky$ which vanish when $x = 0, and, when, x = l$

iii. By considering the characteristic functions of the problem
$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0,$$

Show that $\int_{-1}^{1} P_r(x)P_s(x)dx = 0$

8.0 REFERENCES/FURTHER READING

Earl, A. Coddington (1989). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

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