UNIT 1 TOPOLOGICAL SPACES

CONTENTS

1.0 INTRODUCTION

In your study of metric spaces, you defined a number of key ideas like, limit point, closure of a set, etc. In each case, the definition rests on the notion of a neighbourhood, or, what amounts to the same thing, the notion of an open set. You in turn defined the notions (neighborhood and open set) by using the metric (or distance) in the given space. However, instead of introducing a metric in a given set X, you can go about things differently, by specifying a system of open sets in X with suitable properties. This approach leads to the introduction of the notion of a topological space. Metric spaces are topological spaces of a rather special (although very important) kind.

2.0 OBJECTIVES

At the end of this unit, you shall be:

- able to define a topological space; and
- conversant with some important topological notions.

3.0 Basic Concepts.

3.1 Definitions and Examples

Definition 3.1 Let X be a set. A topology (τ) on X is a collection of subsets of X, satisfying the following properties:

1. The given set X itself and the empty set ϕ are members of τ
2. Any union of members of τ is a member of τ
3. The intersection of two members of τ is a member of τ

The symbolic form of the three axioms are:

1. $X \in \tau$, $\phi \in \tau$
2. $\tau_u \in \tau$ for every $u \in m \Rightarrow \cup \{\tau_u \mid u \in m\} \in \tau$
3. $\tau_1, \tau_2 \in \tau \Rightarrow \tau_1 \cap \tau_2 \in \tau$

Definition 3.2
By a topological space is meant a pair (X, τ), consisting of a set X and a topology τ defined on X. Just as a metric space is a pair consisting of a set X and a metric defined on X, so a topological space is a pair consisting of a set X and a topology defined on X. Thus to specify a topological space, you must specify both a set X and a topology on X. You can equip one and the same set with various different topologies, thereby defining various different topological spaces. In the sequel, you shall omit τ and call only X a topological space provided no confusion arise.

Definition 3.3
The elements of the topology τ on X are called open sets.

Example 3.1 (Sierpinski topology) Let X= {a,b,c} you can define many topologies on X.

For example, you can define

$[_s = \{ \phi, X, \{b\}, \{a, b\}, \{b, c\}\}$

Then τ_s is a topology on X called the sierpinski topology.

Example 3.2 (The Discrete topology). If X is a set, take τ_d to be all possible subsets of X. τ_d is clearly a topology on X, it is called the discrete topology. In the discrete topology, all subsets of X are open. It is the largest topology on X

Example 3.3 (The Indiscrete topology). Let X be a set, and let $\tau_t = \{\emptyset , X\}$. Then τ_t is clearly a topology on X called the indiscrete or trivial topology. It is the smallest topology on X and (X, τ_t) is called the topological space of coalesced points. This is mainly of academic interest.

Example 3.4 (Finite complement topology). Let X be a set, and let τ_f be the collection of all subsets U of X such that X r U is either finite or X, i.e., τ_f is the collection of the form

$\tau_f := \{U \subset X : \text{either } X \setminus U \text{ is finite or } X \setminus U = X \}$.

Then τ_f is a topology of X called the finite complement topology.

Example 3.5 Let X be a set, and let τ_c be the collection of subsets U of X such that X r U is either countable or is X, i.e., τ_c is a collection of the form

$\tau_f := \{U \subset X : \text{either } X$ **r** $U \text{ is at most countable or } X$ **r** $U = X \}$

Then τ_c is a topology on X.

Definition 3.4

Let τ_1 and τ_2 be two topologies on X. Then τ_1 is said to be finer than τ_2 (i.e., τ_2 is coarser than τ_1) if $\tau_1 \supset \tau_2$.

According to definition (3.4) you can observe that if τ is any topology on X, then

$\tau_t \subset \tau \subset \tau_d$ where τ_d and τ_t are as defined in examples (3.4) and (3.3).

Theorem 3.1. The intersection $\tau = \cap \tau_\alpha$, where each τ_α is a topology (where Δ is some indexing set).

Proof. You are required to verify the three (3) axioms of topology of X for $\tau = \cap \tau_\alpha$, given that $\{\tau_\alpha\}\alpha \in \Delta$ is a family of topologies on X
So proceed as follows:

1. since τ_α is a topology on X for each $\alpha \in \Delta$, the ϕ and X are in each τ_α so that ϕ, X $\in \cap \tau_\alpha$

3.2 Basis for Topology

1. Let $\{\cup_i\}_{i \in I}$ be a collection of elements of τ, where I is some indexing set.
Let $\cup = \cup_{i \in I}$. You have to show that $\cup \in \tau$.
But you already have that for each $i \in I$ $\cup_I \in \tau_\alpha$ for fixed $\alpha \in \Delta$. Since τ_α is a topology on X, $\cup = \cup_{i \in} \tau_\alpha$ for $\alpha \in \Delta$. Therefore, by taking intersection over $\alpha \in \Delta$, you have $\cup = \cup\cup_{i \in I}$ where $\cup \in \tau$.
2. To verify axiom (3), it is enough to do it for two sets $\cup_1$ and $\cup_2$ in τ. The result follows by induction on $\cap$
Therefore, take two sets $\cup_1$ and $\cup_2$ in τ and let

$$\cup = \cup_1 \cap \cup_2$$

You have to show that $\cup \in \tau$. But $\cup_1, \cup_2 \in \tau_\alpha$ for each $\alpha \in \Delta$
Thus $\cup = \cup_1 \cap \cup_2 \in \tau_\alpha$ since each τ_α $\alpha \in \Delta$ is a topology on X. Hence,

$$\cup = \cup_1 \cap \cup_2 \in \tau_\alpha$$

For each example in the proceeding section, you were able to specify the topology by describing the entire collection τ of open sets. This is usually difficult in general. In most cases, you will need to specify instead a smaller collection of subsets of X and then define the topology in terms of this collection.

Definition 3.5
(Basis) Let X be a set. A basis $\mathcal{B}$ for a topology on X is a collection Ψ of subsets of X (called basis elements) such that

1. For each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, or equivalently $X = \cup_{B \in \mathcal{B}}$.
2. If $x \in X$, $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. There exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

Definition 3.6

(Topology generated by a Basis). If Ψ satisfies the above two conditions, then we define the topology τ generated by Ψ as follows:

A subset U of X is in τ (i.e., U is open) if for each $x \varepsilon U$, there exists a basis element $B \varepsilon \Psi$ such that $x \epsilon B \subset U$.
That is to say that τ is a collection of the form
$\tau := \{ U \epsilon X : U = \phi$ or if $x \epsilon U$, there exists $B \epsilon B$ such that $x \epsilon B \subset U\}$
You can easily verify that $\boldsymbol{\tau}$ is a topology on X. Note that each basis element is open.

Example 3.6 Let B= {(a, b): a, b ϵ R, a < b}. Then B is a basis for a topology on R called the standard or euclidean topology on R.

Example 3.7 Let $B^{'}$ = {(a, b) : a, b ϵ R, a < b}. Then $B^{'}$ is a basis for a topology on R called the lower limit topology on R.

Example 3.8 Let B= {{x} : x ϵ X}. Then B is a basis for the discrete topology on X.
Proposition 3.1 Let X be a set, and let B be a basis for a topology $\boldsymbol{\tau}$ on X. Then $\boldsymbol{\tau}$ equals the collection of all unions of elements of B.
Proof. Let $(B_i)_{i \epsilon I}$ be a collection of elements of B. Then for each $i \epsilon I$, $B_i \epsilon \tau$ (because each Bi is open). Since τ is a topology. $B_i \epsilon \tau$.
Conversely, let $U \epsilon \tau$, and let $x \epsilon U$. B is a basis for τ implies there exist $B_x \epsilon B$ such that $x \epsilon B_x \subset U$, this implies that $U = \subset_{x\,u} \{x\} \subset_{x\,u} B_x \subset U$

Thus $U = \subset_{x\,u} B_x$, so that U is a union of elements of B

Example 3.9 Let X = {a, b, c, d, e, f }
and

$$\tau^{'} = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}.$$

Then B = {{a}, {c, d}, {b, c, d, e, f} is a basis for $\tau^{'}$ as $B \epsilon \tau^{'}$ and every element of $\tau^{'}$ can be expressed as a union of elements of B.

Note the τ itself is also a basis for τ^0
So far, you have seen that when you are given a basis, you can define a topology. But the following example tells you that you have to be very careful when you have an arbitrary collection of subsets of a set X.

Example 3.10 Let X = {a, b, c} and B = {{a}, {c}, {a, b}, {b, c}}. Then B is not a basis for any topology on X. To see this, suppose that B is a basis for some topology τ. Then τ consists of all unions of sets in B; that is,

$$\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\}.$$

However, τ is not a topology since $\{a, b\} \cap \{b, c\} = \{b\} \epsilon \tau$. So τ does not have property (3) of Definition 3.1. This is a contradiction, and so your supposition is false. Thus % is not a basis for any topology on X.

In view of the above example, the question of interest now is; under what conditions is of a collection & of subsets of X a basis for a topology on X? The answer to this question is provided by the next proposition.
Proposition 3.2 Let X be a topological space. Suppose that & is a collection of open

subsets of X such that for each open set U of X and each x ⊚⊚◌ U, there exists C ⊚⊚◌& such that

x ε C ⊂ U.

Then & is a basis for a topology of X.

When topologies are given by basis, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than the other. One such criterion is the following:

Proposition 3.3 Let Ψ and Ψ be basis for the topologies τ and τ^0 , respectively, on X. Then the following are equivalent:

1. $\tau^{\prime}$ is finer that τ.
2. For each x ϵ X and each basis element B ϵ B containing x, there exists a basis element $B^{\prime}$ ϵ $B^{\prime}$ such that x ϵ $B^{\prime}$ ⊂ B.

Proof. (1) ⇒ (2). Let x ϵ X and B ϵ B such that x ϵ B. You know that B ϵ τ by definition and that τ ⊂ $\tau^{\prime}$ by condition (1); therefore, B ϵ $\tau^{\prime}$. Since $\tau^{\prime}$ is generated by $B^{\prime}$, then there exists an element $B^{\prime}$ ϵ $B^{\prime}$ such that x ϵ $B^{\prime}$ ⊂ B.

(2) ⇒ (1). Given an element U ϵ τ. Your goal is to show that U ϵ $\tau^{\prime}$. So let x ϵ U. Since B generate τ, there is an element B ϵ B such that x ϵ B ⊂ U. By condition (2) there exists $B^{\prime}$ ϵ $B^{\prime}$ such that x ϵ $B^{\prime}$ ⊂ B. Then x ϵ $B^{\prime}$ ⊂ U, so U ϵ $\tau^{\prime}$, by definition. ■

3.2.1 The Metric Topology

One of the most important and frequently used ways of imposing a topology on a set is to define the toplogy in terms of a metric on a set. Topologies given in this way lie at the heart of modern analysis, for example. In this section, you shall be introduce with the metric topology and some of its examples.

Definition 3.7 A metric on a set X is a function d : X × X → R having the following properties:

1. d(x, y) ≥ 0 for all x, y ϵ X; equality holds if and only if x = y
2. d(x, y) = 0 <=> x = y
3. d(x, y) = d(y, x) for all x, y ϵ X
4. d(x, z) ≤ d(x, y) + d(y, z) for all x, y, z ϵ X (Triangle inequality)

Given a metric d on X, (X, d) is a metric space and the number d(x, y) is called the distance between x and y in the metric d.

Definition 3.8 Let (X, d) be a metrtic space. Let x ϵ X and r > 0. The set

$$B_d(x, r) = \{y \in X: d(x, y) < r\}$$

Of all point y ϵ X whose distance from x is less than r is called the open ball centred at x with radius r, otherwise called open-ball centered at x.

Lemma 3.1 Let d be a metric on the set X. Then the collection of all open-balls B_d (x), for x ϵ X and r > 0 is a basis for a topology on X, called the metric topology induced by d.

Proof. The first condition of a basis is trivial since $x \in B(x, \ )$ for any $\ > 0$. Before you check the second condition for a basis, first of all prove the fact that if $y \in B(x, \ )$ for some $x \in X$ and $\ > 0$, there exists $\delta > 0$ such that $B(y, \delta) \subset B(x, \ )$. Define $\delta = \ - d(x, y)$, then by triangle inequality, if $z \in B(y, \delta)$ then $d(x, z) \leq d(x, y) + d(y, z) < \ $. Now to check the second condition for basis, let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$. Choose δ_1 and δ_2 such that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. Let $\delta = \min(\delta_1, \delta_2)$, you have $B(y, \delta) \subset B_1 \cap B_2$. ■

Using what you have just proved, you can rephrase the definition of the metric topology as follows:

Definition 3.9 A set U is open in the metric topology induced by d if and only if for each $x \in U$ there exist $\ > 0$ such that

$$B_d(x, \ ) \subset U.$$

Example 3.11 Given a set X, define

$$\mathbf{d(x, y)} = \begin{cases} 1 \, if \;\; x \neq y \\ 0 \, if \;\; x = y \end{cases}$$

3.2.2 Product Topology

Here, you shall be introduced to the product topology, but a detailed study of this kind of topology will be done in subsequent units.

Let X and Y be topological spaces. There is a standard way of defining a topology on the cartesian product X x Y. We consider this topology now and study some of its properties.

Lemma 3.2. Let X and Y be two topological spaces. Let % be the collection of all sets of the form U x V, where U is an open subset of X and V is an open subset of Y. i.e.,
B := {UxV : U is open in X and V is open in Y}

Then B is basis for a topology on X x Y.

Proof. The first condition is trivial, since $X \times Y$ is itself a basis element. The second condition is almost easy, since the intersection of any two basis element $U_1 \times V_1$ and $U_2 \times V_2$ is another basis element. For

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2),$$

and the later set is a basis element because $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y, respectively.

Definition 3.10 Let X and Y be topological spaces. The Product topology on X × Y is the topology having the collection B as basis.

It is easy to check that B is not a topology itself on X × Y. You may now ask, what if the topologies on X and Y are given by basis? The answer to this question is in what follows.

Theorem 3.2 If B is a basis for the topology on X and C is the basis for the topology on Y, then the collection

$$D = \{B \times C : B \in B \text{ and } C \in C\}$$

is a basis for the topology on X × Y.

Proof. You can use proposition 3.2. Given an open set W of X×Y and a point (x, y) ∈ X×Y of W, by definition of the product topology, there exists a basis element U × V such that (x, y) ∈ U × V ⊂ W. Since B and C are bases for X and Y, respectively, you can choose an element B ∈ B such that x ∈ B ⊂ U and an element C ∈ C such that y ∈ C ⊂ V. So (x, y) ∈ B × C ⊂ U × V ⊂ W. Thus the collection D meets the criterion of proposition 3.2. so D is a basis of X × Y. ■

Example 3.13. You have the standard topology of R. The product topology of this topology with itself is called the Product topology on R™ R = R^2. It has as basis the collection of all products of open sets of R, but the theorem you just proved tells you that the much smaller collection of all products (a, b) ™ (c, d) of open intervals in R will also serve as a basis for the topology of R^2. Each such set can be pictured as the interior of a rectangle in R^2. It is sometimes useful to express the product topology in terms of sub-basis. To do this, we just define certain functions called projections.

Definition 3.11 Let $\pi_1 : X \times Y \to Y$ and let $\pi_2 : X \times Y \to Y$ defined by

$$\pi_1(x, y) = x \text{ and } \pi_2(x, y) = y.$$

The maps π_1 and π_2 are called projection of X ×Y onto its first and second factors, respectively.

The word onto is used because they are surjective (unless one of the spaces X or Y happens to be empty, in which case X × Y is empty and your whole discussion is empty as well).

If U is an open subset of X, then $\pi_1^{-1}(U)$ is precisely the set U × Y, which is open in X × Y. Similarly, if V is open in Y, then $\pi_2^{-1}(V) = X \times V$, which is also open in X × Y. The intersection of these two sets in the set U × V. This fact leads to the following theorem.

Theorem 3.3. The collection

$$S = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on X × Y.

Proof. Let τ denote the product topology on X × Y, let τ^0 be the topology generated by S. Since $S \subset \tau$ then arbitrary unions of finite intersections of elements of S stay in τ. Thus $\tau^0 \subset \tau$. On the other hand, every basis element U × V for the topology τ is a finite intersection of elements of S, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Therefore U × V belongs to τ^0, so $\tau \subset \tau^0$ as well. ∎

3.2.3 The Subspace Topology

Definition 3.12 Let X be a topological space with topology τ. If Y is a subset of X, the collection

$$\tau_Y = \{Y \cap U : U \in \tau\}$$

is a topology on Y, called the subspace topology. With this topology. Y is called a subspace of X; its open sets consists of all intersection of open sets of X with Y.

Lemma 3.3 If B is a basis for the topology on X, the collection

$$B_Y = \{B \cap Y : B \quad B$$

is a basis for the subspace topology in Y.

Proof. Let U be an open set of X and $y \in U \cap Y$, By definition of basis, there exists $B \in B$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from proposition 3.2 that B_Y is a basis for the subspace topology on Y. ∎

When dealing with a space X and a subspace Y of X, you need to be careful when you use the term open set. The question is do you mean an element of the topology of Y or an element of the topology on X? The following definition is useful. If Y is a subspace of X, the set U is open in Y (or open relative to Y) if it belongs to the topology of Y: this implies in particular it is a subspace of Y. There is a special situation in which every open set in Y is also open in X.

Lemma 3.4. Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.

Proof: Since U is open in Y, U = V n Y for some V open in X. Since Y and V are both open in X, so is V n Y.

Proposition 3.4 Let A be a subspace of X and B a subspace of Y. Then the product topology on A x B is the same as the topology A x B inherits as a subspace of X x Y.

3.3 Closed Sets and Limit Points

Now that you have a few examples at hand, you can proceed to see some of the basic concepts associated with topological space. In this section, you shall be introduced to the notion of closed set, interior, closure and limit point of a set.

3.3.1 Closed Sets

Definition 3.13. A subset A of a topological set X is said to be closed if X \ A, the complement of A in X is open.

Example 3.14 The subset [a, b] of R is closed because its complement ||R\[a, b]

is open. Similarly [a, $+\infty$) is closed.

Example 3.15 Consider the following subset of the real line: Y = [01]∪ (2, 3), in the subspace topology. In this space, the set [0, 1] is open, since it is the intersection of the open set $-\frac{1}{2}, \frac{3}{2}$ of R with Y. Similarly, (2, 3) is open as subset of Y. Since [0, 1] and (2, 3) are complement in Y of each other, you can conclude that both are closed as subset of Y.

The collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X.

Theorem 3.4. Let X be a topological space. Then the following conditions hold:

1. Ø and X are closed.
2. Arbitrary intersection of closed sets is closed.
3. Finite unions of closed sets are closed.

Proof: Apply the De Mogan's Laws:
1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

When dealing with subspaces, you need to be very careful in using the term open set. The following theorem is very important.

Theorem 3.5 Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof: Assume that A = C n Y, where C is closed in X, then X r C is open in X, so that (X **r** C) n Y is open in Y, by definition of the subspace topology. But (X r C) n Y = Y **r** A. Hence Y **r** A is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. The set X **r** U is closed in X, and A = Y n (X **r** U), so that A equals the intersection of a closed set of X and Y, as desired.

Note that a set that is closed in the subspace Y may not be closed in X. So the question now is, when is a closed set in a subspace Y closed in the space X ? The next theorem provides

an answer to this question.
Theorem 3.6 Let Y be a subspace of X. If A is closed in Y, and Y is closed in X, then A is closed in X.

3.3.2 Closure and Interior of a Set

Definition 3.14 Let A be a subset of a topological space X. The interior of A denoted by Int or A^o is defined as the union of all open sets contained in A. The closure of A denoted by cl (A) or A^o is defined as the intersection of closed sets containing A.

Clearly, the interior of A is an open set and the closure of A is a closed set; furthermore, If A is OPEN, Then A= int (A); on the other hand, if A is closed, then A = $\bar{A}$

Proposition 3.5 Let Y be a subspace of X; Let A be a subset of Y. Let $\bar{A}$ denote the clusure of A in X. Then the closure of A in Y is $\bar{A} \cap Y$.

Another useful way of describing the closure of a set is given in the following theorem.

Theorem 3.7 Let A be a subset of the topological space X.

1. The $x \in \bar{A}$ if and only if every open set U containing x intersects A.
2. Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof. Consider the statement (a). It is a statement of the form P $\Leftrightarrow$ Q. Transforming each statement to is contrapositive, gives you the logical equivalence (not P) $\Leftrightarrow$ (not Q). Explicitly,

$x \in \bar{A}$ if and only if there exists an open set U containing x that does not intersect A.

In terms of this assertion, the theorem is easy to prove. If x is not in $\bar{A}$, the set X – A is open and contains x and does not intersect A as desired. Conversely, If there exists an open set U containing x which does not intersect A, then X – A is a closed set containing A. By definition of the closure A, the set X – U must contain A; therefore $x \notin$ A. ∎
Part (b) follows from the definiton of basis.

Definition 3.15 Let X be a topological space. Let x ε X and V be a subset of X containing x. V is said to be a neighbourhood of x if there exist and open set U of X such that

$$x \in U \subset V.$$

The collection of all neighbourhoods of x is denoted by **N** (x).

Proposition 3.6 Let X be a topological space and $x \in X$. Then

1. **N** (x) is nonempty;
2. If $V \in$ **N** and $V \subset A$ then $A \in$ **N** (x);
3. A finite intersection of neighbourhoods of x is a neighbourhood of x.

Proposition 3.7 Let X be a topological space. Let U be a subset of X. Then U is open if and only in $U \in \mathbf{N}(x)$ for every $x \in U$.

Lemma 3.5 If A is a subset of a topological space X, then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A. i.e.,

$$x \in \overline{A} \text{ if and only if for all } V \in \mathbf{N}(x),\ V \cap A = \varnothing .$$

Proof. ($\Rightarrow$) Let $x \in \overline{A}$, and let $V \in \mathbf{N}(x)$. Since $V \in \mathbf{N}(x)$, there exist U open such that $x \in U \subset V$. It is enough for you to show that $U \cap A = \varnothing$. Suppose $U \cap A = \varnothing$, it implies that $A \subset U^c$. And U^c is closed since U is open, thus, $\overline{A} \subset U^c$. Which implies that $x \in U^c$, which is a contradiction. Hence, $U \cap A = \varnothing$.

($\Leftarrow$) Assume that for every neighbourhood V_c of x, $V \cap A = \varnothing$. You have to show that

$x \in \overline{A}$. Suppose $x \in \overline{A}$, this implies that $x \in \overline{A}$ which is open (because $\overline{A}$ is closed) and so $\overline{A}^c \in \mathbf{N}(x)$, and by hypothesis, $\overline{A}^c \cap A = \varnothing$. This is a contradiction, hence $x \in A$.

Example 3.16 Let X be the real line R. If $A = (0, 1]$, then $\overline{A} = [0, 1]$, $B = \{1/n : n \geq 1\}$ then $\overline{B} = B \cup \{0\}$. If $C = \{0\} \cup (1, 2)$ then $\overline{C} = \{0\} \cup [1, 2]$, $= \mathbb{R}$.
$\overline{Q}$

Example 3.17 Consider the subspace $Y = (0, 1]$ of the real line R. The set $A = (0, \frac{1}{2})$ is a subset of Y. Its closure in R is the set $[0, \frac{1}{2}]$ and its closure in Y is the set $\overline{A} = [0, \frac{1}{2}] \cap Y = (0, \frac{1}{2}]$.

3.3.3 Limit Points—

Definition 3.16 Let A be subset of a topological set X and let x ε X. x is said to be a limitpoint (or cluster point or point of accumulation) of A if every neighbourhood of x intersects A in some point other than that x itself.

$$x \in X \text{ is a limit point of A if for all } V \in \mathbf{N}(x),\quad V \cap (A \setminus \{x\}) = \varnothing .$$

Or x is a limit point of A if x belongs to the closure of $A \setminus \{x\}$. The point x may lie in A or not.

4.0 CONCLUSION

Theorem 3.8 Let A be a subset of the topological space X. Let A be the set of all limit points of A. Then $\overline{A} = A \cup A^0$.

Proof. Clearly, $A \cup A^0 \subset \overline{A}$. To prove the reverse inclusion, let $x \in \overline{A}$. If x happens to be in A, it is trivial that $x \in A \cup A^0$. Suppose that $x \notin A$. Since $x \in A^0$, this implies that every neighbourhood U of x intersects A. Because $x \notin A$, the set U intersects A in a point different from x. Then $x \in A^0$, so that $x \in A \cup A^0$ as desired. ■

Corollary 3.1 A subset of a topological space is closed if and only if it contains all its limit points.

Proof. The set A is closed if and only if $A = \overline{A}$, and the later holds if and only if $A^0 \subset A$. ■

In this unit, you have been introduced to the meaning and examples of topological spaces and some basic concepts of topological spaces such as basis for a topology, closed set, open sets, interior of a set, closure of a set, neighbourhood of a set and limit point of a set. You have seen some examples and proved some results.

5.0 SUMMARY

Having gone through this unit, you now know that;

(i) a topology defined on a set X is a collection τ of subsets of X satisfying
(a) X and Ø are in τ,
(b) arbitrary unions of elements of τ are in τ,
(c) finite intersections of elements of τ are in τ.

(ii) a topological space is a pair (X, τ) consisting of a set X and a topology τ defined on it.

(iii) the elements of a topology on X are called open sets.

(iv) if τ_1 and τ_2 are topologies defined on X, then τ_1 is said to be finer that τ_2 if $\tau_2 \subset \tau_1$. In other words you say that τ_2 is coarser than τ_1.

(v) an arbitrary intersection of topologies is also a topology.

(vi) a basis for a topology τ on X is a collection B of subsets of X (i.e., basis elements) such that

(a) for each X, there exist B ε B such that x ε B, or equivalently, $X = \cup_{B \varepsilon B}$
(b) if x ε X and B_1, B_2 ε B such that x n B_1 n B_2, there exists B_3 ε B such that x ε $B_3 \subset B_1$ n B_2.

(viii) the topology generated by a basis B is given by

$$\tau = \{U \in X : U = \phi \text{ or if } x \in U, \text{ there exists } B \in \mathfrak{B} \text{ such that } x \in B \subset U\}$$

(viii) The collection

$$B := \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a basis for the product topology on X × Y.

(ix) The collection

$$S = \{\pi^{-1}(U) : U \text{ is open in } X\} \cup \{\pi^{-1}(V) : V \text{ is o}\rceil\, \}\text{ı in } Y$$

is a subbasis for the product topology on X × Y. Where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection maps defined on X × Y by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

(x) if Y is a subset of a topological space (X, τ), the collection

$$\tau_Y = \{Y \cap U : U \in \tau\}$$

is a topology on Y, called the subspace topology. Y is called a subspace of X, its open sets consists of all intersection of open sets of X with Y.

(xi) A subset A of a topological space X is said to be closed in X if X **r** A, (the complement of A in X) is open.

(xii) if X is a topological space, then
(a) Ø and X are closed.

(b) an arbitrary intersection of closed sets is closed.
(c) a finite union of closed sets is closed.

(xiii) if Y is a subspace of X, then a set A is closed in Y if and only if it equals the intersection of a closed set in X with Y.

$^{\circ}$ is the union of all open sets contained in A, while the closure of $\overline{A}$ denoted by A is the intersection of all closed sets contained in A.

(xv) if V is a subset of a topological space X and x ε X such that x ε V, then V is called a neighbourhood of x if there exists an open set U of X such that

$$x \varepsilon U \subset V.$$

(xvi) 1 (x) denotes the collection of all neighbourhoods of x.

(xvii) if A is a subset of a topological space X, an element x of X is called a limit point of A if for all V ε N (x), V n (A **r** {x}) = Ø.

(xviii) a subset of a topological space is closed if and only if it contains all its limit point.

6.0 TUTOR MARKED ASSIGNMENTS

Exercise 6.1

1. In the following, answer true or false.

   (a) The collection

   $$\tau_\infty = \{U : X \mathbf{r} U \text{ is infinite or empty or all } X\}$$

   is a topology in X?

   (b) The union $\bigcup \tau_\alpha$ of a family $\{\tau_\alpha\}$ of topology on X is a topology on X.

   (c) The countable collection

   $$\mathsf{B} = \{(a, b) : a < b, a, b \in Q\}$$

   is a basis for a topology on R.

   (d) If A is a subset of a topological space X, and suppose that for each x ∈ A, there exists an open set U such that x ∈ U ⊂ A, then A is an open set in X.

2. Let R be with the standard topology and let A ⊂ R. Then A is open in R if there exist an interval I such that I ⊂ A. For a, b ∈ R, which of the following forms is is the interval I

   (a) I = (a, b)

   (b) I = (a, b]

   (c) I = [a, b)

   (d) I = [a, b]

3. If τ is a topology on a set X, which of the following is not true about τ?

(a) Finite union of elements of τ is in τ.

(b) Finite intersection of elements of τ are in τ. (c)

The empty set $\varnothing$ and the whole set X are in τ. (d)

Arbitrary intersection of elements of τ are in τ.

4. Answer true or false. The collection

$$B = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is

(a) a topology on the product space $X \times Y$.

(b) a basis for a topology on the product space $X \times Y$.

5. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps defined by

$$\pi_1(x, y) = x \text{ and } \pi_2(x, y) = y.$$

$$S = \{\pi_1^{-1}(U) | U \text{ open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ open in } Y\}$$

is ________________ for the product topology on $X \times Y$.

(a) a collection of open sets

(b) a basis

(c) a subbasis

(d) a topology

6. Let R be endowed with the standard topology. Consider the set $Y = [-1, 1]$ as a subspace of R. Which of the following sets are open in Y?

$$A = \left\{x : \frac{1}{2} < |x| < 1\right\}$$

$$B = \left\{x : \frac{1}{2} < |x| \leq 1\right\}$$

$$C = \left\{x : \frac{1}{2} \leq |x| < 1\right\}$$

$$D = \left\{x : \frac{1}{2} \leq |x| \leq 1\right\}$$

(a) A, B and C only

(b) A only

(c) B and C only.

(d) D only.

7. With the standard topology of R. which of the sets in question 6 above are open in R?

(a) A, B and C only

(b) A only

(c) B, C and D only.

(d) D only.

8. Let R be endowed with the standard topology. Consider the set $Y = [-1, 1]$ as a subspace of R. Which of the following sets are closed in Y ?

$$A = \left\{ x : \frac{1}{2} < |x| < 1 \right\}$$

$$B = \left\{ x : \frac{1}{2} < |x| \leq 1 \right\}$$

$$C = \left\{ x : \frac{1}{2} \leq |x| < 1 \right\}$$

$$D = \left\{ x : \frac{1}{2} \leq |x| \leq 1 \right\}$$

(a) A, B, C and D.

(b) B and C only

(c) B, C and D only.

(d) D only.

9. With the standard topology of R. which of the sets in question 8 above are closed in R?

(a) A, B and C only

(b) B, C and D only

(c) B and C only.

(d) D only.

10. If $A \subset X$, a topological space, then the boundary of A, denoted by ∂A of Bd A by:

$$\partial A = \bar{A} \cap \overline{X \setminus A}.$$

The folowing are true;

1. $\mathring{A}$ and ∂A are disjoint, and $\bar{A} = \mathring{A} \cup \partial A$.
2. $\partial A =$ set if and only if A is both open and closed.
3. U is open if and only if $\partial U = \bar{U} \setminus U$.

Justify.

11. Hence or otherwise compute the boundary and interior of each of the following subsets of R^2

(a) $A = \{(x, y) : y = 0\}$

(b) $B = \{(x, y) : x > 0 \text{ and } y = 0\}$

(c) $C = A \cup B$.

(d) $D = \{(x, x) : x \text{ is rational}\}$

12. If R, the real line is endowed with the indiscrete topology. Let $A = [0, 1)$. What is $\bar{A}$?

(a) $[0, 1]$

(b) R

(c) $[0, 1)$

(d) $\varnothing$

[Hint: Use theorem 3.7]

13. If R, the real line is endowed with the usual metric topology, and let $A = (0, 1)$. What is ∂A?

(a) R

(b) $[0, 1]$

(c) $\{0, 1\}$

(d) $(0, 1]$