MODULE 1

UNIT 1 LINEAR INTEGRAL EQUATION: PRELIMINARY CONCEPTS

CONTENTS

1.0 INTRODUCTION

In integral equations, an unknown function which is the subject seeking a solution always appears under an integral sign. These equations bear a close kinship with differential equations suggesting that a differential equation can be formulated as an integral equation and vice-versa.

The analytical method remains the standard method of solving integral equations, however, where the analytical method fails; the equation can be solved numerically.

Let us commence with two common problems to illustrate the basic concepts of linear integral equations; loaded elastic string and the shop stocking problem.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

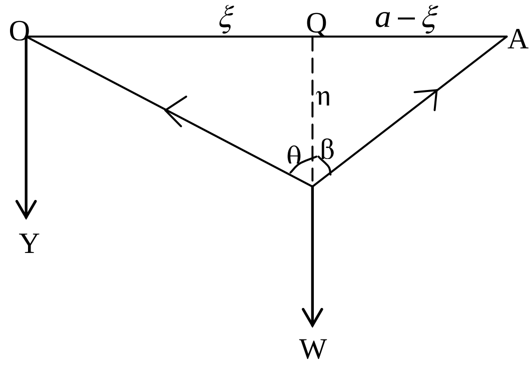
- explain the basic concepts underlying linear integral equations;
- investigate the equations which describe the displacement of a loaded elastic sting; and
- treat the shop stocking problem.

3.0 MAIN CONTENT

3.1 Linear Integral Equation: Preliminary Concepts

Let us take a look at some problems, the types of which we encounter every day and which give rise to integral equation.

3.1.1 A Loaded Elastic String



Consider a weightless elastic string as shown in the above figure, stretched between two horizontal points O and A and suppose that a weight W is hung from the elastic string and that in equilibrium the position of the weight is at a distance ξ from O and at a depth Y below OA. If W is small compared to the initial tension T in the string, it can be assumed that the tension of the string remains T during the further stretching. The vertical resolution of forces gives the equilibrium equation $T\,(\eta/\xi) + T\,(\eta/(a-\xi)) - W = O$

Where $AO = a$ (1.1)

The drop Y due to a weight W situated a distance ξ along the string from O is given by

$$Y = W(a-\xi)\xi/Ta \tag{1.2}$$

The drop Y in the string at a distance x from O is given by

$$Y = x\,y/\xi, \qquad 0 \le x \le \xi \tag{1.3}$$
$$y = (a-x)\eta/(a-\xi), \; \xi \le x \le a \tag{1.4}$$

Eliminating y, these two results can be written in the form

$$y = W\,G(x,\xi)/T \tag{1.5}$$

where

$$G\,(x,\xi) = x(a-\xi)/a, \qquad 0 \le x \le \xi$$
$$= \xi\,(a-x)/a, \qquad \xi \le x \le a \tag{1.6}$$

Suppose now that the string is loaded continuously with a weight distribution $W(x)$ per unit length, the elementary displacement at the point distance x from O, due to the weight distribution over $\xi \le x \le \xi + \partial\xi$ is

$$\partial y \;=\; W(\xi)\partial\xi \;\; G(x,\xi)/T$$

$$0 \le x, \xi \le a \quad (1.7)$$

On integrating, displacement due to the complete weight distribution is given by

$$y(x) \;=\; T^{-1}\int_0^a G(x,\xi)\,W(\xi)d\xi, \qquad 0 \le x \le a \tag{1.8}$$

Thus, the displacement of the string is given in terms of the weight distribution. However, if we are given the displacement of the string, what is the weight distribution?

In this case, we can sew site to equation. (1.8) the form

$$y(x) \;= (Ta)^{-1}\left[x\int_0^x (a-\xi)\,W(\xi) + (a-x)\int_x^a \xi \;\, W(\xi)D\xi\right] \tag{1.9}$$

Different this twice, we obtain

$$y''(x) \;\; = (Ta)^{-1}\,W(x)$$

i.e.

$$W(x) \;\; = \; Ta\; y''(x) \tag{1.10}$$

3.1.2 The Shop Stocking Problem

A shop starts selling some goods. It is found that a proportion $K(t)$ remains unsold at time t after the shop has purchased the goods. It is required to find the stock at which the shop should purchase the goods so that the stock of the goods in the shop remains constant (all processes are deemed to be continuous).

Suppose that the shop commences business in the goods by purchasing an amount A of the goods at zero time, and buys at a rate $Q(t)$ subsequently. Over the time interval

$$K(t-\tau)\,Q(\tau)\,d\tau \tag{1.11}$$

Thus, the amount of goods remaining unsold at time t, and which was bought up to that time, is given by. $AK(t) \; + \; \int_0^t K(t-\tau)\,Q(\tau)\,d\tau$ (1.12)

This is the total stock of the shop and is to remain constant at its initial value and so

$$AK(t) \; + \; \int_0^t K(t-\tau)\,Q(\tau)\,d\tau \tag{1.13}$$

And the required stocking rate $Q(t)$ is the solution of this integral eqn.

4.0 CONCLUSION

You have learnt the processes involved in the two illustrative problems. It is easy to formulate similar solutions for a vast array of problems.

5.0 SUMMARY

The two problems presented demonstrate how to formulate and derive an integral equation for a suitably structured problem. It also demonstrates the process of solving the integral equation developed.

- Apart from the Loaded Elastic String and the Shop Stocking Problem, can you make a list of 5 different types of problems which can be solved using integral equation?

- A transport company distributed workshops within a metropolis which receives and repairs its broken down vehicles. The workshop manager discovers that he must always reroute a *Y (t)*% of his workshop allocation of vehicles to alternative location every day as he cannot accommodate them in his workshop overnight, and he calls you in to tell him the optimum number of requests for repairs he should entertain every day such that the workshop is 100% utilised when all related processes are assumed to be continuous. Formulate an integral equation to help the workshop manager.

7.0 REFERENCES/FURTHER READING

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UNIT 2 CONVERSIONS OF ORDINARY DIFFERENTIAL EQUATIONS INTO INTEGRAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

There are many ordinary differential equations which can be converted into corresponding integral equations and we shall proceed to study how these transformations can be carried out; particularly in the classical case of the Sturm Lowville problems and a host of others illustrative of this transformation process.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- convert ordinary differential equations into integral equations;
- transform Sturm- Liouville problems to integral equations; and
- work through a series of examples of transformations and conversions, and their solutions.

3.0 MAIN CONTENT

3.1 Conversion of Ordinary Differential Equations into Integral Equations

$$y^{11}(x) + a_1(x)\, y^1(x) + a_2(x)\, y(x) = f(x) \tag{1.14}$$

with the initial condition,

$$y(0) = y_0, \quad y^1(0) = y_1 \tag{1.15}$$

Let $\psi(x) = y^{11}(x)$ (1.16)

Then, $y^1(x) = \int_0^x \psi(u)\, du + y_1$ (1.17)

$$y(x) = \int_0^x (x-u)\ \psi(u)du + y_1 x + y_0 \tag{1.18}$$

Substituting the relations 1.16 to 1.18 into the differential equation, it follows that

$$\psi(x) + \int_0^x [a_1(x) + a_2(x)(x-u)]\psi(u)\,du$$

$$= f(x) - y_1 a_1(x) - y_1 x\, a_2(x) - y_0 a_2(x) \tag{1.19}$$

Equation (1.19) can be written in the form

$$\psi(x) + \int_0^x K(x,u)\psi(u)\,du = g(x) \tag{1.20}$$

Which is an integral equation for $\psi(x)$

Example 1.1

Form the integral equation corresponding to

$$y^{11} + 2xy^1 + y = 0, \quad y(0) = 1, \quad y^1(0) = 0$$

Solution

Let $y_x^{11} = \psi(x), \quad y^1 = \int_0^x \psi(u)du$

$$y = \int_0^x (x-u)\psi(u)du + 1$$

Thus, $\psi(x) + 2x\int_0^x \psi(u)du + \int_0^x (x-u)\psi(u)\,du + 1 = 0$

i.e. $\psi(x) + \int_0^x (3x-u)\psi(u) + 1 = 0$

3.2 Transformation of Sturm - Liouville Problems to Integral Equation

A problem which is associated with an expression of the form

$$Ly = \frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) - q(x)y, \quad x_1 \le x \le x_2 \tag{1.21}$$

and boundary condition of the form

$$a_1 y(x_1) + b_1 y^1(x_1) = 0 \tag{1.22}$$

$$a_2 y(x_2) + b_2\, y^1(x_2) = 0$$

is said to be of Sturm-Louville type.

There are two problems which are of interest here, namely:

$$Ly = f(x) \qquad x_1 \le x \le x_2 \tag{1.23}$$

and

$$Ly + \lambda r(x) y = 0 \qquad x_1 \le x \le x_2 \tag{1.24}$$

are continuous in the interval $x_1 \le x \le x_2$, and in addition $P(x)$ has a continuous derivative and does not vanish.

The differential equation (1.23) corresponds to a displacement y caused by some forcing function f, and the differential equation (1.24) forms together with the boundary condition, an Eigenvalue problem.

Suppose that Q_1, Q_2 are solutions of the equation $Ly = 0$

with $a_1Q_1(x_1) + b_1 Q_1(x_1) = 0$

$$a_2Q_2(x_2) + b_2Q_2{}^1(x_2) = 0 \tag{1.25}$$

then,

$$\begin{aligned} 0 &= Q_2 \, LQ_1 - Q_1 \, LQ_2 \\ &= Q_2 \frac{d}{dx}\left(P\frac{d\varepsilon_1}{dx}\right) - Q_1 \frac{d}{dx}\left(P\frac{dd_2}{dx}\right) \\ &= \frac{d}{dx}\left(P\left(Q_2\frac{dQ_1}{dx} - Q_1\frac{dQ_2}{dx}\right)\right) \end{aligned}$$

Thus,

$$P\left(Q_2\frac{dQ_1}{dx} - Q_1\frac{dQ_2}{dx}\right) = \text{constant} \tag{1.26}$$

Using the method of variation of parameters, look for a solution of the form

$$y(x) = z_1(x)\,Q_1(x) + z_2(x)\,Q_2(x) \tag{1.27}$$

where z_1 and z_2 are to be determined.

Thus,

$$y^1 = z^1{}_1\, Q_1 + z_2^1\, Q_2 + z_1Q_1^1 + z_2Q_2^1 \tag{1.28}$$

Let $z_1{}^1 Q_1 + z_2^1 Q_2 = 0,$ so that

$$Ly = \frac{d}{dx}\left[P(x)\left(z_1(x)\,Q_1^1\,(x) + z_1(x)\,Q_2^1(x)\right)\right]$$

$$-\,q(x)\,(z)(z_1(x)\,Q_1(x) + z_2(x)\,Q_2(x))$$

$$= \quad P\left(z_1^1\,Q_1^1 + z_2^1\,Q_2^1\right) \tag{1.29}$$

Since $LQ_1 = LQ_2 = 0$

Thus, z_1 and z_2 are given by the solutions of equations

$$z_1^1\,Q_1 \;+ z_2^1\,Q_2 \;=\; 0 \tag{1.30}$$

$$P\left(z_1^1\,Q_1^1 + z_2^1\,Q_2^1\right) = \;f(x) \tag{1.31}$$

Whence, $z_1{}^1 = \dfrac{fQ_2}{P\left(Q_2\,Q_1{}^1 - Q_1Q_2^1\right)}$ $\qquad z_2{}^1 \;=\; \dfrac{-\,fQ_1}{P\left(Q_2Q_1{}^1 - Q_1Q_2^1\right)}$ (1.32)

The denominator in these two expressions is constant by (1.26) and by a suitable scaling of ϕ_1 and ϕ_2 may be taken as -1.

Thus,

$$z_1^1 = -\;fQ_2, \qquad z_2^1 \;=\; fQ_1 \tag{1.33}$$

It follows that

$$z_1(x) \;=\; -\int_x^x Q_2(u)\;f(u)\,du \tag{1.34}$$

$$z_2(x) \;=\; \int_p^x Q_1(u)\;f(u)\,du \tag{1.35}$$

where the unspecified limits of integration are the equivalent of the arbitrary constants of integration and are determined by the necessity of y satisfying the boundary condition.

Now,

$$a_1 y + b_1 y^1 = a_1\left(z_1 Q_1 + z_2 Q_2\right) + b_1\left(z_1 Q_1{}^1 + z_2 Q_2^1\right) \tag{1.36}$$

Since $z_1^1\,Q + z_2{}^1\,Q_2 \;=\; 0$

Also $a_1 Q_1(x_1) \;+ b_1\;Q_1^1(x_1) = 0$ (1.37)

Hence,

$$0 = a_1 y(x_1) + b_1 y^1(x_1) = z_2(x_1) \left(a_1 Q_2(x_1) + b_1 Q_2{}^1(x_1)\right) \quad (1.38)$$

First let us assume that neither Q_1 nor Q_2 satisfies both boundary condition, hence, it follows that $z_2(x_1) = 0$ and so

$$z_2(x) = \int_{x_1}^{x} Q_1(u)\, f(u)\, du \quad (1.39)$$

Similarly,

$$a_2 y + b_2 y^1 = a_2 \left(z_1 Q_1\ z_2 Q_2\right) + b_2 \left(z_1^1 Q_1 + z_1 Q_1^1 + z_2^1 Q_2 + z_2 Q_2{}^1\right)$$

$$= a_2 \left(z_1 Q_1 + z_2 Q_2\right) + b_2 \left(z_1 Q_1^1 + z_2 Q_2^1\right)$$

$$= z_2 \left(a_2 Q_2 + b_2 Q_2^1\right) + z_1 \left(a_2 Q_1 + b_2 Q_1{}^1\right)$$

Since $a_2\ Q_2(x_2) + b_2 Q^1(x_2) = 0,$ we have

$$0 = a_2 y(x_2) + b_2 y^1(x_2) = z_1(x_2) \left(a_2 Q_1(x_2) + b_2 Q_1{}^1(x_2)\right)$$

Thus, it follows that $z_1(x_2) = 0$ and so

$$z_1(x) = -\int_{x_2}^{x} Q_2(u)\, f(u) du$$

$$= \int_{x}^{x_2} Q_2(u)\, f(u) du \quad (1.39)$$

Hence

$$y(x) = z_1(x) Q_1(x) + z_2(x) Q_2(x)$$

$$= Q_1(x) \int_{x}^{x_2} Q_2(u)\, f(u) du + Q_2(x) \int_{x_1}^{x} Q_1(u) f(u) du$$

$$y(x) = \int_{x_1}^{x_2} G(x, u)\, f(u) du \quad (1.40)$$

where

$$G(x, u) = Q_1(u) Q_2(x) \quad x_1 \le u \le x \quad (1.41)$$

The quantity $G(x, u)$ is termed the Green's fin associated with the operate L and the boundary condition specified.

We would see that the Eigenvalue problem (1.24) defined and the boundary condition (1.25) can be reformulated as the integral equation

$$y(x) + \lambda \int_{x_1}^{x_2} G(x, u)\, r(u)\, y(u) du = 0 \qquad (1.42)$$

by just replacing $f(x)$ by $\lambda r(x)\, y(x)$.

Let us now consider the case where one of the solutions ϕ_1 and Q_2 of $Ly = 0$ do satisfy both boundary condition while the other will not satisfy either boundary condition. Then, following the provided argument, if follows that

$$y(x) = Q(x) \int_x^x \psi(u) + (u) du \quad + \psi(x) \int_x^\beta Q(u) f(u) du \qquad (1.43)$$

where x and β are arbitrary and here $\psi(x)$ does not satisfy either boundary conditions.

Since both y and Q satisfy the boundary condition, if follows that

$$0 = a_1 y(x_1) + b_1 y^1(x_1) = \left(a_1 \psi_1(x_1) + b_1 \psi_1{}^1(x_1)\right) \int_{x_1}^\beta Q(u) f(u) du \qquad (1.44)$$

$$0 = a_2 y(x_2) + b_2 y^1(x_2) = (a_2 \psi_2(x_2)) \int_{x_2}^\beta Q(u)\, f(u) du \qquad (1.45)$$

$\psi(x)$ does not satisfy either boundary condition and so if follows that from (1.44) $\beta = x,$ and from (1.45) we have

$$\int_{x_1}^{x_2} Q(u)\, f(u) du = 0 \qquad (1.46)$$

and the solution is only possible when this relation exists between f and Q. Thus, the integral equation formulation becomes

$$y = A\, Q(x) + \int_{x_1}^{x_2} G(x, u)\, f(u)\, du \qquad (1.47)$$

Wher +e $A = \int_x^{x_1} \psi(u)\, f(u)\, du$ is an arbitrary constant and

$$G(x, u) = Q(u) \psi(x) \qquad x_1 \le u \le x$$

$$= Q(x) \psi(u) \quad x \le u \le x_2 \qquad (1.48)$$

Example 1.2

Find an integral equation formulation for the problem defined by

$$\frac{d^2 y}{dx^2} + 4y = f(x) \qquad 0 \le x \le \pi/4, \quad y = 0 \text{ at } x = 0, \text{ and } y = 0 \text{ at } x = \pi/4$$

Solution

The solutions of $\frac{d^2y}{dx^2} + 4y = 0$ which satisfy the boundary condition at $x = 0$ and $x = \pi/4$ are $Sin2x$ and $Cos2x$ respectively.

Neither satisfies both boundary conditions.

Let, $y = w\sin 2x + z\cos 2x$

$\therefore \quad y^1 = w^1 \sin 2x + z^1 \cos 2x + 2w\cos 2x\ 2z \sin 2x$

$\therefore \quad = 2w\cos x - 2z\sin 2x$ if $w^1 \sin x + z^1 \cos 2x = 0$

$\therefore \quad y^{11} = 2w^1 \cos 2x - 2z^1 \sin 2x - 4w\sin 2x - 4z\cos 2x$

Thus,

$$y^{11} + 4y = f$$

becomes

$$2w^1 \cos 2x - 2z\sin 2x = f$$

whence,

$$z^1 = -\frac{1}{2} f \sin 2x, \qquad w^1 = \frac{1}{2} f \cos 2x$$

Thus,

$$z(x) = -\frac{1}{2}\int_\alpha^x f(u)\sin 2u du \text{ and } w(x) = \frac{1}{2}\int_\alpha^x f(u)\cos 2u du$$

$$\therefore \quad y = \frac{\sin 2x}{2}\int_\beta^x f(u)\cos 2u du - \frac{\cos 2x}{2}\int_\alpha^x f(u)s 2u du$$

Now $y = 0$ at $x = 0$, so that

$$0 = 0 - \frac{1}{2}\int_\alpha^0 f(u)\sin 2u du.$$

$$\therefore \quad \alpha = 0$$

Also, $y = 0$ at $x = \pi/4$, so that

$$0 = \frac{1}{2}\int_{\beta}^{\pi/4} f(u)\cos 2u du - 0$$

$\therefore \beta = \pi/4$. Thus,

$$y = -\frac{1}{2}\sin 2x\int_{x}^{\pi/4} f(u)\cos 2u du - \frac{1}{2}\cos 2x\int_{0}^{x} f(u)\sin 2u du$$

$$= \int_{0}^{\pi/4} G(x,u)\, f(u)du$$

where $G(x,u) = \frac{-1}{2}\cos 2x\sin 2u \quad 0 \le u \le x$

$$= \frac{-1}{2}\sin 2x\cos 2u \quad x \le u \le \pi/4.$$

Example 1.3

Transform the problem defined by

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

when $y = 0$ at $x = 0$ and $y^1 = 0$ at $x = 1$ into integral equation form.

Solution

The solution to this problem is

$$y = \sin\frac{(2n-1)\pi x}{2}, \quad \lambda = \left[\frac{(2n-1)\pi}{2}\right]^2 \qquad n = 1, 2, 3, \cdots$$

The two solutions $\frac{d^2y}{dx^2} = 0$ which satisfy the boundary conditions are respectively $y = x$ and $y = 1$. (neither satisfies both b.c)

Following through the usual process, if follows that the solution of

$\frac{d^2y}{dx^2} = f(x)$ under the boundary condition specified is

$$y = x\int_{1}^{x} f(u)\,du + \int_{x}^{0} u f(u)du$$

and so the integral formulation is

$$y(x) = \lambda\int_0^1 K(x,u)\,y(u)du$$

where

$$K(x,u) = x \qquad 0 \le x \le u \qquad \begin{cases} 1 \le u \le x \\ x \le u \le 0 \end{cases}$$
$$= u \qquad u \le x \le 1$$

Example 1.4

Transform the problem by $\dfrac{d^2y}{dx^2} + y = f(x)$

and the boundary condition $y = 0$ at $x = 0$ and $x = \pi$ into integral equation form and indicate what condition must be satisfied by $f(x)$.

Solution

Now $\sin x$ satisfies the equation $\dfrac{d^2y}{dx^2} + y = 0$ and both boundary condition

The second solution of the differential equation $\dfrac{d^2y}{dx^2} + y = 0$ is $\cos x$, and this satisfies neither boundary conditions

Let $y = z\sin x + w\cos x$

Following the same process, it follows that

$$y = \sin x\int^x \cos u\ f(u)du + \cos x\int_x \sin u\ f(u)du$$

Now y is to vanish at $x = 0$, and so the limit of integration on the second integral is zero y must also vanish $x = \pi$ and it follows therefore, that

$$y(\pi) \quad \cos\pi\int_\pi^0 \sin u\ f(u)\,du \quad = 0$$

Thus, for a solution to be possible

$$\int_0^\pi \sin u\ f(u)du = 0$$

and $y(x) = A\sin x + \int_0^\pi G(x,u)\ f(u)du$

where A is arbitrary and

$$G(x,u) = -\sin u\cos x \qquad 0 \le u \le u$$

$$= -\sin x \cos u \qquad x \le u \le \pi.$$

4.0 CONCLUSION

A Sturm–Lowville differential equation with boundary conditions may be solved by a variety of numerical methods on most occasions; however, there are situations where it becomes necessary to carry out intermediate calculations.

5.0 SUMMARY

Ordinary differential equations can be transformed into integral equations.

6.0 TUTOR-MARKED ASSIGNMENT

1. Transform the problem defined by *y'' – Ky = 0* when y=0 at x=2 and y'=0 at x=4 into integral equation?

2. A Sturm-Lowville type problem can be associated with an expression of the form

$$Ly = \frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) - q(x)y, \quad x_1 \le x \le x_2 \tag{1.21}$$

   Write down the form of the second boundary condition when the first is of this form.

   $a_1 y(x_1) + b_1 y^1(x_1) = 0$?

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UNIT 3 CLASSIFICATION OF LINEAR INTEGRAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

Integral equations are classified according to Limits of integration, placement of unknown function and nature of known function. These result in Fredholm and Volterra equations on the one hand, and integral equations on the other hand. Finally, the homogeneous and non-homogeneous falling into the last class – making a total of eight distinct classes of integral equations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- classify Linear integral equations; and
- find approximate solutions for integral equations.

3.0 MAIN CONTENT

3.1 Classification of Linear Integral Equation

Let $K(x, y)$ be a function of two variables x and y defined and let $f(x)$ and $Q(x)$ be two functions of the variable x continuous in the interval $a \le x \le b$, which are connected by the functional equation

$$f(x) = Q(x) - \lambda \int K(x, y)\, Q(y) dy \qquad (1.49)$$

The functional equation (1.49) is called a linear integral equation of the 2nd kind with the kernel $K(x, y)$. In this equation, every continuous function $Q(x)$ is transformed into another continuous function $f(x)$; the transformation is linear, since to $c_1 Q_1 + c_2 Q_2$, there corresponds the analogous combination $c_1 f_1 + c_2 f_2$.

If the function $f(x)$ vanishes identically, we are dealing with a homogenous integral equation. If a homogenous equation possesses a solution other than the trivial solution

$Q = 0,$ the solution may be multiplied by an arbitrary constant factor and may therefore, be assumed normalised.

If $Q_1, Q_2, \cdots, Q_n$ are solutions of the homogenous equation, then, all linear combination $C_1\ Q_1 + \cdots + C_n\ Q_n$ are solutions.

It can also be proved that linearly independent solutions of the same homogenous internal equation are orthornormal. A value λ for which the homogenous equation possesses non-vanishing solutions is called an Eigen function of the kernel for the Eigenvalue λ. Their number is finite for each Eigenvalue.

The integral equation $\int_a^b K(x, y)\, Q(y)\, dy = f(x)$ (1.50)

1st kind. The integral equation

$$Q(x) = \lambda \int_a^b K(x, y) Q(y) dy + f(x), \quad a \le x \le b \tag{1.51}$$

is termed a Fredholm equation of the 2nd kind.

If $K(x, y) = 0 \quad y > x,$ (1.52)

the kernel is said to be of Volterra type.

The integral equation

$$\int_a^x K(x, y)\, Q(y) dy = f(x) \quad a \le x \tag{1.53}$$

is termed a Volterra integral equation of the 1st Kind.

If $K(x, y) = K(y - x),$ the kernel is said to be of convolution form.

The integral equation

$$Q(x) = \lambda \int_0^x K(x, y)\, Q(y) dy + f(x) \quad a \le x \tag{1.54}$$

is termed a Volterra integral equation of the 2nd kind.

In general, it is a Volterra integral equation of the integral equation of the 2nd kind.

If we differentiate equation (1.53) w.r.t $x,$ it follows that

$$K(x, x)\ Q(x) + \int_a^x \frac{\partial K(x, y)}{\partial x} Q(y)\, dy = f^1(x) \tag{1.55}$$

If $K(x, x)$ is non-zero, it is possible to divide through by it, and it is clear that it is an associated Volterra integral equation of the 2nd kind.

The kernel is said to be symmetric.

if $K(x, y) = -K(y, x)$

The kernel is said to be anti-symmetric

if $K(x, y) = \overline{K}(y, x)$

The kernel is said to be Hermitian

if $K(x, y) = \overline{K}(y, x)$

3.2 Approximate Solutions

We split the interval into n equal sub-interval, and suppose that we may write approximately

$$K(x, y) = K_{rs}\left(\frac{r-1}{n} \leq x \leq \frac{r}{n}, \frac{s-1}{n} \leq y \leq \frac{s}{n}\right)$$

where Krs are constants.

Similarly, when we write

$$f(x) = f_r \quad \left(\frac{r-1}{n} \leq x \frac{r}{n}\right)$$

the equation (1.54) becomes

$$Q(x) = f_r + \lambda \sum_{s=1}^{n} K_{rs} \int_{\frac{s-1}{n}}^{s/n} Q(y)dy \tag{1.56}$$

$$\frac{r-1}{n} \leq x \leq \frac{r}{m}$$

This shows that Q also will be a step function taking the values Q_r, say.

Equation (1.56) becomes

$$Q_r - \frac{\lambda}{n} \sum_{s=1}^{n} K_{rs} Q_s = f_r \tag{1.57}$$

Let K be the $n \times n$ matrix with elements $\frac{K_{rs}}{n}$ and let Q be n vectors, then, we have

$$(I - \lambda K)Q = f \tag{1.58}$$

The system has thus been reduced approximately to a set of linear algebraic equations. For these, the theory is well-known and a computational solution is straight forward.

In a sense, the solution of (1.54) may be regarded as the limit of (1.57) as $n \to \infty$.

Exercises

Solve the equation

(i) $Q(x) \ - 2\int_0^1 xy\, Q(y)dy \ = \ x^2$

(ii) $Q(x) \ + \ \int_0^1 e^{x-t} Q(t)dt \ = \ x \qquad Q(0) = 0$

approximately at the parts. $x = 0, \frac{1}{2}, 1$ and $y = 0, \frac{1}{2}, 1$

Compare your results with the exact solution in case (ii).

4.0 CONCLUSION

Linear integral equations can be classified into several groups and sub-groups such as: Fredholm, Hermitian, Volterra integral equation and those integral equations which are either symmetric or anti-symmetric.

5.0 SUMMARY

Linear integral equations can be classified according to their common characteristics.

6.0 TUTOR-MARKED ASSIGNMENT

1. In how many ways can integral equations be classified?
2. What type of integral equations has a fixed (constant) limit of integration?
3. Distinguish a Volterra type of integral equation from a Fredholm integral equation.
4. A homogeneous equation is identically non-zero. True or False?

7.0 REFERENCES/FURTHER READING

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