MODULE 2

UNIT 1 S2 VOLTERRA INTEGRAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

Volterra integral equations have integration limit which include the variable as opposed to the Fredholm integral in which the integration limits are constants.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- recognise volterra integral equations;
- comprehend that there are the three types of volterra integral equations; and
- arrive at the resolvent kernel of a volterra equation.

3.0 MAIN CONTENT

Volterra integrals are characterised by the limit of integration being one variable and of which there are three types. A common solution to Volterra integrals is to employ the formalism known as the Resolvent.

3.1 Volterra Integral Equations

A kernel $K(x, y)$ is said to be of Volterra type if $K(x, y) = 0, \quad y > x$ (2.1)

There are three types of Volterra integral equations.

These are:

(i) The equation of the first type.

$$f(x) = \int_0^x K(x, y)\, Q(y)\, dy \qquad (2.2)$$

(ii) The equation of the second type.

$$Q(x) = \lambda \int_0^x K(x, y)\, Q(y)\, dy + f(x) \qquad (2.3)$$

(iii) The homogenous equation of the second type.

$$Q(x) = \lambda \int_0^x K(x, y)\, Q(y)\, dy \qquad (2.4)$$

The following properties arise:

(i) It is necessary for consistency in the equation of the first kind i.e. $f(0) = 0$

(ii) Any solution to the equation of the second kind cannot be correct unless $Q(0) = f(0)$

(iii) If K is non-singular, there are no Eigenvalue and Eigen functions associated with the homogenous equation (2.4)

(iv) The equation of the first type can be differentiated to give the equivalent equation

$$K(x, x)\, Q(x) + \int_0^x \frac{\partial K(x, y)}{\partial x} Q(y) = f^1(x) \qquad (2.5)$$

Example 2.1

Solve the integral equation

$$Q(x) = 3\int_0^x \cos(x - y)\, Q(y)\, du + e^x$$

Solution

Here $Q(0) = f(0) = 1$

Differentiating w.r.t x, it follows that

$$Q^1(x) = 3\, Q(x) - 3\int_c^x \sin(x - y)\, Q(y)\, dy + e^x$$

Thus, $Q^1(0) = 3Q(0) + 1 \quad 4$

Differentiating w.r.t x again, we have

$$Q^{11}(x) = 3\ Q^{1}(x) - 3\int_0^x \cos(x-y)\,Q(y)dy + e^x$$

$$= 3\,Q^{1}(x) - Q(x) + 2e^x$$

This equation can simply be solved thus:

$$Q^{11} - 3\ Q^{1}(x) + Q(x) = 2e^x$$

Consider the homogenous equation

$$Q^{11} - 3\ Q^{1} + Q = 0$$

Let $Q = e^{mx} \Rightarrow m^2 - 3m + 1 = 0, \quad m = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Example 2.2

Solve the integral equation

$$Q(x) = x + 1 + \int_0^x (1 + 2(x-y))\,d(y)\,dy$$

Solution

Differentiating once, it follows that $(Q(0) = f(0) = 1)$

$$Q^{1}(x) = + Q(x) + 2\int_0^x Q(y)dy$$

$$Q^{1}(0) = 1 + Q(0) = 2$$

Differentiating again, we have

$$Q^{11}(x) = Q^{1}(x) + 2\,Q(x)$$

i.e. $Q^{11} - Q^{1} - 2\,Q = 0$ Let $Q = e^{mx}$, $m^2 - m - 2 = 0$ $(m+1)(m+2) = 0$

$\Rightarrow\ m = -$ or 2

$$Q = A\,e^{-x} + Be^{2x}$$

3.2 Resolvent Kernel of Volterra Equation

Let us consider the equation:

$$Q(x) - \lambda\int_0^x K(x, y)\,Q(y)\,dy = f(x) \tag{2.6}$$

We can set about the solution by guessing that at least for small x the integral term will be small. First approximation is then

$$Q_0(x) = f(x) \tag{2.7}$$

So that $\int_0^x K(x, y)\,Q(y) \div \int_0^x K(x, y)\,Q_0(y)\,dy$

$$= \int_0^x K(x, y)\,f(y)\,dy \tag{2.8}$$

The second approximation $Q_1(x)$, is then

$$Q_1(x) = f(x) + \lambda\int_0^x K(x, y)\,f_0(y)\,dy$$

$$= f(x) + \lambda\int_0^x K(x, y)\,f_0(y)\,dy \tag{2.9}$$

Repeating the argument, we obtain a sequence of approximations.

$$Q_n(x) = f(x) + \lambda\int_0^x K(x, y)\,Q_{n-1}(y)\,dy \tag{2.10}$$

Write equation (2.10) in the form

$$Q_n = f + \int K\,Q_{n-1} \tag{2.11}$$

So that $Q_{n-1} = f + \lambda\int K\,Q_{n-2}$ (2.12)

Therefore, $Q_n - Q_{n-1} = \lambda\int K\,(Q_{n-1} - Q_{n-2})$ (2.13)

Now set $\psi_0 = f = Q_0$ and $\lambda^n\psi_n = Q_n - Q_{n-1}$ (2.14)

then,

$$\lambda^n\,\psi_n = \lambda\int K\,\lambda^{n-1}\psi_{n-1}$$

i.e.

$$\psi_n = \int K\psi_{n-1} \qquad n \geq 1 \tag{2.15}$$

Now, $Q_0(x) \;=\; f(x)$

$$\psi_1(x) \;=\; \int_0^x K(x, y)\, f(y)\, dy$$

$$\psi_2(x) \;=\; \int_0^x K(x, y) \quad \psi_1(y)dy = \int_0^x K(x, z)\psi_1(z)dz$$

$$= \int_0^x K(x, z)\, dz \int_0^z K(z, y)\, f(y)\, dy$$

$$= \int_0^x f(y)dy \int_0^x K(x, z)\, K(z, y)\, dz$$

$$= \int_0^x f(y)\, K_2\,(x, y) \qquad (2.16)$$

where $K_2 \;=\; \int_y^x (x, z)\, K(z, y)\, dz$

By repetition of the argument, we have

$$\psi_n(x) \;=\; \int_0^x K_n\,(x, y)\, f(y)dy \qquad (2.17)$$

Where $K_1(x, y) \;=\; K(x, y)$ and

$$K_{n+1}(x, y) \;= \int_y^x K(x, z)\, K_n\,(z, y)\, dz \qquad (2.18)$$

Also, from equation $\lambda^n \psi_n \;=\; Q_n - Q_{n-1}$, so that

$$\sum_{r=0}^{n} \lambda^r \psi_r \;=\; (Q_n - Q_{n-1}) + (e_{n-1}\, Q_{n-2}) + \cdots + (Q_1 - Q_0) + Q_0$$

$$=\; Q_n \qquad (2.19)$$

$$\therefore \qquad Q_n \quad \sum_{r=0}^{n} \lambda^r \psi_r \qquad (2.20)$$

By considering equation (2.20), (2.17), we have

$$Q_n = \;\; f(x) \;+ \int_0^x \left(\sum_{r=1}^{n} \lambda^r\; K_r\,(x, y) \right) + (y)\, dy$$

$$(\Psi_0 = Q_0) \qquad (2.21)$$

Thus, it is plausible to suppose that

$$Q(x) = \lim_{n\to\infty} Q_n(x)$$

$$= \quad f(x) \quad + \quad \int_0^x \left[\sum_{r=1}^{\infty} K_r(x, u)\right] + f(y)dy \qquad (2.22)$$

$$= \quad f(x) \quad -\lambda \int_0^x R(x, y, \lambda)\, f(y)dy \qquad (2.23)$$

where $R(r, y) = -\sum_{r=0}^{\infty} \lambda^r K_{r+1}(x, y).$ (2.24)

The function R is called the Resolvent kernel.

Let us now determine the conditions under which the power series on the right hand side of equation is convergent.

Suppose that over $0 \le x, \quad y \le l, |K(x, y)| \le K$

Then,

$$|K_2(x, y)| = \left| \int_y^x K(x, z)\, K(z, y)\, dz \right|$$

$$\le \quad K^2(x-y) \quad = \quad (x-y)K^2 \quad x \ge y \qquad (2.25)$$

Also $K_2(x, y) = 0, \qquad y \ge x$

Similarly,

$$|K_3(u, y)| \quad = \quad \left| \int_y^x K_2(x, z)\, K(z, y)\, dz \right|$$

$$\le K^3 \int_y^x (x-z)dz = \frac{1}{2} K^3 (x-y)^2 \quad x \ge y \qquad (2.26)$$

and $K_3(x, y) = 0, \qquad x \le y$

Proceeding in this way, it follows that:

$$|\, K_n(x, y)| \le \frac{1}{(n-1)}1 \;\; K^n (x-y)^{n-1} \quad x \ge y \qquad (2.27)$$

$$= \quad 0 \qquad x \le y$$

Thus, the series $\lambda^n \; K_n\,(x,\, y)$ is dominated by the series with the n[th] term

$$\frac{\lambda^n}{(n-1)!}\;\; K^n\,(x-y)^{n-1} \tag{2.28}$$

now $|x-y| \le 2l,$ and so the later series is dominated by the series with n[th] term

$$\frac{\lambda^n K}{(n-1)!}\,(2lK)^{n-1} \tag{2.29}$$

This is the typical term of an exponential series and so it follows that the series 2.23 for $R\,(x,\, y,\, \lambda)$ always converge.

The uniqueness of the solution follows easily because, if $Q_A(x), \;\; Q_B\,(x)$ are both solution, then,

$$Q_A\,(x) \;\; - Q_A(x), \;\; = \;\; \int_0^x K\,(x,\, y)\,(Q_A(y)\,Q_B\;(y)dy) \tag{2.30}$$

Since the resolvent kernel series converges for all values of λ is the original kernel n bounded.

This is equivalent to saying that there is no Eigenvalue. Thus, $Q_n(x) \; = \; Q_B(x)$

Example 2.3

Solve the integral equation

$$Q(x,\, y) \;\; = \;\; f\,(x,\, y) \; + \; \int_0^x \int_0^y \exp\,(x-u+y+v)\,Q(u,v)\;\; d_u d_v$$

$$K_1\,(x,\, y;\, u,\, v) \; = \; \exp\,(x-u+y-v)$$

$$K_2\,(x,\, y;\, u,\, v) \; = \; \int_u^x \int_v^u K\big(x,\, y,\, x^1,\, y^1\big) \;\;\; K\,\big(x^1,\, y^1,\, u,\, v\big)\,dx^1 dy^1$$

$$= \;\; \exp\,(x-u+y-v) \;\; \int_u^x dx^1 \int_v^y dy^1 = (x-u)\,(y-v)\exp\;\; (x-u+y-v)$$

Similarly,

$$K_3\,(x,\, y;\, u,\, v) \; = \;\; \int_u^x \int_v^y \; \exp\,(x-u+y-v)\,\big[\big(x-x^1\big)\big(y-y^1\big)\big]\,dx^1 dy^1$$

$$= \;\; \exp\,(x-u+y-v)\int_v^y \Big(xx^1 - \tfrac{x^{12}}{2}\Big)\,\Big|_u^x \,\big(y-y^1\big)\,dy^1$$

$$= \exp\ (x-u+y-v)\left[\left(x^2 - \frac{x^2}{2} - xu + \frac{u^2}{2}\right)\left(y^2 - \frac{y^2}{2} - vy + \frac{v^2}{2}\right)\right]$$

$$= \quad \frac{1}{2^2}(x-u)^2\ (y-v)^2\ \ \exp\ (x-u+y-v)$$

Hence,

$$K_n\ (x,\ y;\ u,\ v) \quad = \quad \frac{(x-u)^{n-1}(y-v)^{n-1}}{[(u-1)\,!]^2} \exp\ (x-u+y-v)$$

and so

$$R(x,\ y;\ u,\ v,) \quad = \quad -\sum_{u-1}^{\infty} K_n\ (x,\ y;\ u,\ v)$$

$$= \quad -\exp\ (x-u+y-v)\sum_{u=0}^{\infty} \frac{(x-u)^u\ (y-v)^x}{(u\,!)^2}$$

The solution is therefore, given as

$$\mathrm{Q}\,(\mathrm{x},\ \mathrm{y}) \quad = \quad f\,(x,\ y) - \int_o^x \int_o^y R(x,\ y;\ u,\ v)\ \ f\,(u,\ v)\,dudv$$

4.0 CONCLUSION

Certain properties arise as a consequence of the three types of Volterra integrals.

5.0 SUMMARY

There are three types of Volterra integral equations, and can be solved using the Resolvent kernel.

6.0 TUTOR-MARKED ASSIGNMENTS

1. How many different type of Volterra Integrals are there. 1, 2, 3 or 4?
2. Which of these three is a Volterra integral equation of the first type?

$$f(x) = \int_0^x K\,(x,\ y)\,Q(y)dy$$

$$Q(x) \quad = \quad \lambda \quad \int_0^x K\,(x,\ y)\,Q(y)dy$$

$$Q(x) \quad = \quad \lambda \int_0^x K\,(x,\ y)\ \ Q(y)dy\ \ + f\,(x)$$

3. And which is a Volterra integral of the third type?

7.0 REFERENCES/FURTHER READING

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Whittaker, E. T. & Watson, G. N. *A Course of Modern Analysis*. Cambridge Mathematical Library.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007*). "Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

UNIT 2 CONVOLUTION TYPE KERNELS

CONTENTS

1.0 INTRODUCTION

Laplace transformation serves as a powerful tool in the solving of integral equations. Convolution, the inverse of Laplace transformation, is necessary to transform the solution back to the originating domain.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- describe how convolution type kernels of the Volterra integral can be solved using Laplace transform;
- solve Fredholm equations; and
- identify a Neumann series.

3.0 MAIN CONTENT

3.1 Convolution Type Kernels

If the kernel of the Volterra integral is of the form $K(x-y)$, the equation is said to be of convolution type and may be solved by using the Laplace transform. The method of solution depends upon the well known result in Laplace transform that:

$$\int_o^\infty e^{-px} \int_o^x F(x-y)\, G(y) dy\, dx$$

$$= \int_o^\infty e^{-px} F(x) dx \int_o^\infty e^{-px} G(x)\, dx \quad (2.36)$$

The term $\int_o^x F(x-y)\, G(y)\, du = \int_o^x F(y)\, G(x-y)\, dy$ (2.37)

is the convolution, (faltung) of the two functions $F(x)$ and $G(x)$.

Let us denote $\int_o^\infty e^{-px} G(x)\,dx,$ the Laplace transform of $G(x)$ by $\overline{G}$.

Consider the integral equation of the first kind.

$$f(x) = \int_0^x K(x-y)\, d(y) dy \tag{2.38}$$

On taking the Laplace transform, it follows that,

$$\overline{F} = \overline{K}\ \overline{Q} \tag{2.39}$$

Thus, $\overline{Q} = \bar{f}/\bar{k}$, (2.40)

provided the transforms exist.

The solution is found by finding the inverse transform of $\overline{Q}$. It is also possible to solve the inhomogeneous Volterra equation of the 2nd kind with the convolution kernels in exactly the same way.

The equation

$Q(x) = f(x) + \int_o^x k(x-y)\, Q(y)\, dy$ transforms into

$\overline{Q} = \overline{F} + \overline{K}\ \overline{Q}$

where $\overline{Q} = \left(1-\overline{K}\right)^{-1} \bar{f}$ (2.41)

and $Q(x)$ may be found.

Example 2.5

Solve the integral equation

$$\int_o^x \sin x\,(x-y)\, d(y)\, dy = 1-\cos\beta x$$

Note that the equation in self-consistent

Taking the Laplace transform, we have

$$\frac{\alpha}{p^2+\alpha^2}\ \overline{Q} = \frac{1}{p} - \frac{p}{p^2+p^2} = \frac{p^2+\beta^2-p^2}{p\left(p^2+\beta^2\right)}$$

Thus, $\overline{Q} = \dfrac{\beta^2\left(p^2+\alpha^2\right)}{\alpha\, p\left(p^2+\beta^2\right)} = \dfrac{\alpha}{p} + \left(\dfrac{\beta^2-\alpha^2}{\alpha}\right) \dfrac{p}{p^2+\alpha^2}$

Therefore, $Q = \alpha + \left(\frac{\beta^2 - \alpha^2}{\alpha}\right)\cos\beta x$

Example 2.6

Solve the integral equation

$$\int_o^x Q(x-y)\left[Q(y) - 2\sin ay\right]dy = x\cos ax$$

Solution

Taking the transform if follows that

$$\overline{Q}\left\{\overline{Q} - \frac{2a}{p^2 + a^2}\right\} = \frac{p^2 - a^2}{\left(p^2 + a^2\right)^2}$$

$$(\textbf{NB})\int\left(x^2 \; f(x)\right) = (-1)^u \frac{d^u}{dp^n}\,|f\,(f))$$

Thus, $\overline{Q} = \sin ax \pm \cos ax$

are the two possible solutions

Example 2.7

Solve the integral equation

$$Q(x) = x^3 + \int_o^x e^{3(x-y)}\,Q\,(y)\,dy$$

Solution

It follows that

$$\overline{Q} = \frac{3!}{p4} + \frac{1}{p-3}\,\overline{Q}$$

$$\therefore \quad \overline{Q} = \left[1 - \frac{1}{p-3}\right]^{-1}\frac{3!}{p4}$$

Hence, $\overline{Q} = \frac{p-3}{p-4}\cdot\frac{3!}{p4} = \frac{3!}{p4}\left(1 + \frac{1}{p-4}\right)$

$$= \frac{3\,!}{p4} + \frac{3!}{p4(p-4)}$$

$$\therefore \quad Q = x^3 + \int_0^x e^{4(x-y)} y^3 \, dy$$

Example 2.8

Solve the integer-differential equation

$$Q^{11}(x) + \int_o^x e^{2(x-y)} Q^1(y) dy = 1$$

where $Q(o) = o, \quad Q^1(o) = o$

Solution

Taking the Laplace transforms, it follows that

$$P^2\overline{Q} + \frac{P\,\overline{Q}}{P-2} = \frac{1}{P} \qquad \left(\overline{Q} = \frac{p-2}{p^2(p-1)^2}\right)$$

and $\overline{Q} = \dfrac{1}{p^2(p-1)^2} = \dfrac{1}{(p-1)^2} - \dfrac{2}{p-1} + \dfrac{1}{p^2} + \dfrac{2}{p}$

Hence,

$$Q(x) = xe^x - 2e^x + x + 2$$

Cossection

$$\overline{Q}\left[p^2 + \frac{p}{p-2}\right] = \frac{1}{p}$$

i.e. $\overline{Q}\left[\dfrac{p(p-1)^2}{p-2}\right] = \dfrac{1}{p}$

$$\Rightarrow \quad \overline{Q} = \frac{p-2}{p^2(p-1)^2} = \frac{-3}{p} - \frac{2}{p^2} + \frac{3}{p-1} - \frac{-1}{(p-1)^2}$$

$$\therefore \quad Q(x) = -3 - 2x + 3e^x - x \; e^x$$

[Reuse partial fraction]

NB: $\dfrac{p-2}{p^2(p-1)^2} \equiv \dfrac{A}{p} + \dfrac{B}{p^2} + \dfrac{C}{p-1} + \dfrac{D}{(p-1)^2}$

$$\Rightarrow \quad p-2 \equiv Ap^3 - 2p^2A + A\,p + Bp^2 - 2Bp + B + cp^3 - cp^2 + Dp^2$$

3.2 Fredholm Equations

The Volterra equations considered are a special case of the equation.

$$Q(x) - \lambda \int_0^1 k(x,y)\, Q(y)\, dy = f(x) \quad (o \le x \le) \tag{3.1}$$

Evidently, the special case is where $k(x, y) = o$ for $y \ge x$

We shall take the interval $(0, 1)$ as standard and for simplicity write the integrals without the limit.

Put $$Q(x) = f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \tag{3.2}$$

where $\psi_1(x) = \int k(x, y)\ f(y)\, dy$

$$\psi_2(x) = \int k(x, y)\psi_1(y)\, dy$$

$$= \int k(x, y)\, dy \int k(y, z)\, f(z)\, dz$$

$$= \int k_2(x, y)\, f(y)\, dy$$

and $$K_2(x, y) = \int K(x, z) \quad K(z, y) dz$$

$$\psi_u(x) = \int K_u(x, y)\, f(y)\, dy$$

where

$$K_u(x, y) = \int K_{uy}(x, z)\ k(z, y)\, dz$$

The series (3.2) is called the Neumann series just as we consider the series for the resolvent kernel.

$$-R(x, y; \lambda) = K(x, y) + \lambda\ k_2(x, y) + \tag{3.3}$$

This series may be proved convergent for a certain sample of values of λ under a variety of conditions. We consider one set of these conditions.

3.3 Lemma 3.1

Suppose $K(x, y)$ is continuous and

$$\sup_{\substack{o \le x \le 1 \\ o \le y \le 1}} |K(x, y)| = M$$

Then, the series (3.3) is uniformly convergent for $|\lambda| < M^{-1}$. It is continuous and the series may be integrated term by term. Also, $R(x, y; \lambda)$ is for each (x, y) an analytic function of the complex variable λ inside $|\lambda| < M^{-1}$.

Proof

We have

$$|K_2(x, y)| = \left| \int_o^2 K(x, x) K(z, y)\, dz \right|$$

$$\leq \sup | K(x, z) K(z, y)|$$

$$\leq \sup|K(x, z)| \sup |K(z, y)| \leq M^2$$

By repeating this, we get

$$|K_n(x, y)| \leq M^n \tag{3.4}$$

Then, the series (3.3) is dominated by $\sum \lambda^n M^n$. The result follows as before by Weierstrass M. test in region $|\lambda M| < 1$. The analyticity is obvious since we are considering each (x, y) the given series $\sum a_n \lambda_n$ where $a_n = K_n(x, y)$. The radius of convergence is not less than M^{-1}. Note that in this case we have only proved convergence for $|\lambda| < M^{-1}$, whereas the Volterra equations are true for all λ finite.

Example 3.1

Consider the integral equation:

$$Q(x) - \lambda \int_o^1 Q(y) dy = f(x)$$

In this case, $K(x, y) = 1$ and $K_n(x, y) = 1$

Thus, $R(x, y; \lambda) = -\sum_{r=o}^{\infty} \lambda^r = \dfrac{1}{\lambda - 1}$

Also, $\sup|K(x, y)| = 1$. Since $\dfrac{1}{\lambda - 1}$ has a pole at 1, the result may not in general be extended to smaller M

If $A = \int_o^1 Q(x)\, dx$, and integrate over $(0, 1)$, the equation

$$Q(x) - \lambda Q(y)\, dy = f(x)$$

$$\therefore \quad A(1-\lambda) = \int_0^1 f(x)\,dx \quad \Rightarrow A = \frac{1}{1-\lambda}\int_0^1 f(x)\,dx$$

Suppose first that $\lambda \neq 1$. Then,

$$Q(x) = f(x) + \lambda A = f(x) + \frac{\lambda}{1-\lambda}\int_o^1 f(x)dx$$

The equation had thus a unique solution.

Suppose on the other hand that $\lambda = 1$.

Then, from the equation $A(1-\lambda) = \int_o^1 f(x)\,dx$ the original equation will only have a solution if $\int_o^1 f(x)\,dx = 0$.

If f does not satisfy this condition and $\lambda = 1$, the equation has an infinite number of solutions $Q(x) = f(x) + c$ where c is a constant and $\lambda = 1$ if an Eigenvalue with corresponding Eigenfunction $Q =$ constant.

Theorem 3.1

Suppose K is continuous in the square $\begin{matrix} S : o \le x \le 1 \\ o \le y \le 1 \end{matrix}$ and set $\sup_s |K| = M$.

The resolvent kernel R is given by

$$-R(x, y; \lambda) = \sum_{r=0}^{\infty} \lambda^r K_{r+1}(x, y) \quad (3.5)$$

Where the series is uniformly convergent for $|\lambda| < m^{-1}$

R is continuous and the series may be integrated term by term. In the domain $\oiint$ where $\oiint$ is analytic. The following relation holds

$$K(x, y) + R(x, y; \lambda) = \lambda \int K(x, z)\, R(z, y; \lambda)\, dz$$

$$= \lambda \int R(x, z; \lambda)\; K(z, y)\, dz$$

Suppose that f is integrable, then, the unique solution for $\lambda \in \oiint$ of (3.1) is

$$Q(x) = f(x) - \lambda \int_0^1 R(x, y; \lambda)\, f(y)dy$$

4.0 CONCLUSION

Convolution type integrals may be solved by the use of Laplace transform provided the transform exists.

5.0 SUMMARY

It is possible to determine if a Volterra integral is of the convolution type and then solve it using the method of Laplace where the final solution is found by finding the inverse transform. This applies also to the inhomogeneous Volterra equation of the 2nd kind which convolution kernels can be solved in exactly the same way.

6.0 TUTOR-MARKED ASSIGNMENT

1. State the name of the integral equation in which the integration limits are constants and do not include the variable?
2. What is the relationship between F(x), G(x) and this term?

$$\int_o^x F(x-y)\,G(y)\,du = \int_o^x F(y)\,G(x-y)\,dy$$

7.0 REFERENCES/FURTHER READING

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Whittaker, E. T. & Watson, G. N. *A Course of Modern Analysis*. Cambridge Mathematical Library.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W. H., Teukolsky, S. A., Vetterling, W. T. & Flannery, B. P. (2007). *"Chapter 19. Integral Equations and Inverse Theory"*. *Numerical Recipes: The Art of Scientific Computing.* (3rd ed.). New York: Cambridge University Press.