# MODULE 3



## UNIT 1 FREDHOLM EQUATIONS WITH DEGENERATE KERNELS

**CONTENTS**



## 1.0 INTRODUCTION

Fredholm integral equations are integral equations in which the integration limits are constants which do not include the variable; and whose solution gives rise to Fredholm theory, the study of Fredholm kernels and Fredholm operators.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- solve Fredholm equations with degenerate kernels; and
- derive the general method of solution of Fredholm equations.

## 3.0 MAIN CONTENT

### 3.1 Fredholm Equations with Degenerate Kernels

Consider the Kernel of the form:

$$K(x, y) = \sum_{p=1}^{n} a_p(x) b_p(y) \tag{3.6}$$

where $x$ is finite, and the $a_r$ and $b_r$ form linearly independent sets. A kernel of this character is termed a degenerate kernel.

Also, consider the integral equation of the first kind

$$f(x) = \int K(x, y)\,Q(y)dy$$

$$= \sum_{p=1}^{n} a_p(x)\int b_p(y)\,Q(y)\,dy \qquad (3.7)$$

1. We note that no solution exist unless $f(x)$ can be written in the form $\sum_{p=1}^{n} f_p\ a_p(x)$ (3.8)

   This is essential for the equation to be self-consistent.

2. The solution is indefinite by any function $\psi(y)$ which is orthogonal to all the $b_p(y)$ over the range of integration.

**Example 3.2**

The integral equation

$\exp(2x) = \int_{\pi}^{\pi} \sin(x+y)\,\phi(y)dy \qquad o \le x \le \pi$ is not self-consistent and so does not have a solution.

This is because

$$\int_{\pi}^{\pi} \sin(x+y)\ \phi(y)\,dy = \sin x \int_{o}^{\pi} \cos y\,\phi(y)dy$$

$$+ \cos x \int_{\pi}^{\pi} \sin y\,\phi(y)dy$$

which is a of form $A\sin x + B\cos x$

## 3.2 The General Method of Solution

Look for a solution of the form

$$\phi(y) = \sum_{q=1}^{n} \phi_q\ b_q(y) \qquad (3.9)$$

If it exists, it will be a solution and if it is possible to add $\psi(y)$ to it. The solution proceeds as follows in the integral equation.

$$f(x) = \lambda\int K(x, y)\,\phi(y)\,dy \qquad (3.10)$$

$$\sum_{p=1}^{n} f_p \; a_p(x) \quad = \quad \lambda \sum_{p=1}^{n} a_p(x) \int b_p(y) \sum \phi_q b_q(y) dy \tag{3.11}$$

$$= \quad \sum_{p=1}^{n} a_p(x) \sum_{q=1}^{n} \beta_{pq} \; \phi_1 \tag{3.12}$$

Where $\beta_{pq} \quad = \quad \lambda \int b_p(y) \, b_q(y) \, dy$ (3.13)

and so the $\phi_s$ are defined by

$$f_p \quad = \quad \sum_{q=1}^{n} \beta_{pq} \; \phi_q \qquad 1 \leq p \leq n \tag{3.14}$$

Since the $b_p$ are linearly independent, the determinant $|\beta_{pq}|$ does not vanish and the $\phi_q$ can be found uniquely. Also, $\psi(y)$ in such that

$$\int \psi(y) \, K(x, y) dy \quad = \quad 0 \tag{3.15}$$

**Example 3.3**

Consider the solution of the integral equation

$$3\sin x + 2\cos x \quad = \quad \int_{-\pi}^{\pi} \sin\,(x + y)\, \phi(y) dy \qquad -\pi \leq x \leq \pi$$

Now $\sin\,(x + y) \; = \; \sin x \cos y + \sin y \cos x$

and so there is consistency

Note also that $\int_{-\pi}^{\pi} \cos y \cos my \; dy \quad = \begin{cases} o \; if \quad m \neq 1 \\ \pi \; if \quad m = 1 \end{cases}$

$$\int_{-\pi}^{\pi} \cos y \sin my \; dy \quad = \quad \begin{cases} o \;\; if \;\; m \neq 1 \\ \pi \; if \;\; m = 1 \end{cases}$$

$$\int_{\pi}^{\pi} \sin y \cos my \; dy \quad = \quad \begin{cases} 0 \;\; if \quad m \;\neq\; 1 \\ \pi \;\; if \;\; m = 1 \end{cases} \quad \int_{-\pi}^{\pi} \quad \sin y \sin my \; dy = \begin{cases} o \;\; if \quad m \neq 1 \\ \pi \; if \;\; m = 1 \end{cases}$$

Hence, the integral equation in indefinite by a quantity of the form

$$\psi(y) \quad = \quad C_o \quad + \quad \sum_{n=2}^{\infty} \left[C_n \cos ny + dn \sin ny\right]$$

Since $\int_{-\pi}^{\pi} \psi(y) \sin(x+y)\, dy = o$

Now, look for a solution of the form

$$\phi(y) = A\cos y + B\sin y$$

$$\therefore \int_{-\pi}^{\pi} \sin(x+y)\,\phi(y)dy = \sin x \int_{-\pi}^{\pi} \cos y\,(A\cos y + B\sin y)dy$$

$$+ \cos x \int_{-\pi}^{\pi} \sin y\ (A\cos y + B\sin y)dy$$

$$= \prod A \sin x + \pi B \cos x$$

$$\equiv 3\sin x + 2\cos x$$

Thus, $A = {}^{3}\!/_{\pi}$ and $B = {}^{2}\!/_{\pi}$

$$\therefore\ d(y) = (3\cos y + 2\sin y)/\pi$$

Note that the process is similar to the idea of finding the particular integral and complementary function in differential equation theory.

The solution

$$\phi(y) = (3\cos y + 2\sin y)/\pi$$

May be termed a particular solution while the $\psi(y)$ a complementary function.

## 4.0 CONCLUSION

Fredholm equations can be solved by applying the method of degenerate kernel.

## 5.0 SUMMARY

Fredholm integral equations have limits which are constants and not the variable as in the Volterra integral equations.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. What kind of kernel is of the form $K(x, y) = \sum_{p=1}^{n} a_p(x) b_p(y)$ where $x$ is finite, and $a_r$ and $b_r$ form linearly independent sets?

2. Why does $\exp(2x) = \int_{\pi}^{\pi} \sin(x+y)\,\phi(y)dy \qquad o \le x \le \pi$ not have a solution

## 7.0 REFERENCES/FURTHER READING

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

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# UNIT 2 EIGENFUNCTIONS AND EIGENVECTORS

## CONTENTS



## 1.0 INTRODUCTION

Many homogeneous linear integral equations may be viewed as the continuum limit of Eigenvalue equation.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- work with Eigenfunctions and Eigenvectors; and
- prove that symmetric and continuous Kernels that are not identically zero possess at least one Eigenvalue.

## 3.0 MAIN CONTENT

### 3.1 Eigenfunctions and Eigenvectors

Eigenfunction and Eigenvectors associated with the equation:

$$\phi(x) = \lambda \int \sum_{p=1}^{n} a_p(x) b_p(y)\, d(y) dy \tag{3.16}$$

can be found as follows

Rewrite (3.16) in the form

$$\mu\, \phi(y) = \int \sum_{p=1}^{n} a_p(x) b_p(y) \phi(y)\, dy \tag{3.17}$$

This equation satisfied by any function $\phi(y)$ such that

$$\int b_p(y) \phi(y)\, dy = o \tag{3.18}$$

and $\mu = o$, but in general, we shall ignore such functions, any Eigenfunction must be of the form

$$\Phi(x) \;=\; \sum_{p=1}^{n} \phi_p \; a_p(x) \qquad (3.19)$$

Thus $\sum_{p-1}^{n} \phi_p \; a_p(x) \;=\; \lambda \sum_{p=1}^{n} a_p(x) \int b_p(y) \sum_{q=1}^{n} \phi_q \; a_q(y) dy \qquad (3.20)$

Whence

$$\phi_p \;=\; \sum_{q=1}^{n} \phi_q \; K_{pq} \qquad (3.21)$$

$$K_{pq} \;=\; \lambda \int b_p(y)\, a_q(y)\, dy \qquad (3.22)$$

**Example 3.4**

Find the Eigenvalue and Eigenfunction of the system defined by:

$$\phi(x) \;=\; \lambda \int_o^1 (1 + xt)\, \phi(t)\, dt \qquad o \le x \le 1$$

**Solution**

Let $\Phi(x) \;=\; \phi_o + \phi_1 x \;= \lambda \int_o^1 (1 + xt)(\phi_o + \phi_1 t) dt$

$$= \;\; \lambda\left(\phi_o + \frac{\phi_1}{2}\right) + \lambda\left(\frac{\phi_o}{2} \;+\; \frac{\phi_1}{3}\right) \; x$$

Whence (equating coefficients)

$$(\lambda - 1)\, \phi_o + \lambda \,\frac{\phi_1}{2} \;\;=\;\; o$$

$$\frac{\lambda\, \phi_o}{2} \;+\; \left(\frac{\lambda}{3} - 1\right)\phi_1 \;\;=\;\; o$$

Thus, $(\lambda - 1)\left(\lambda/3 - 1\right) \;\;=\;\; \lambda^2/4$

$$\lambda \;\;=\;\; 8 \pm \sqrt{52}$$

and $\quad \phi_1 \,, \phi_o \;\;=\;\; -\left(7 \pm \sqrt{52}\right): \;\; \left(4 \pm \sqrt{13}\right)$

Consider now the solution of the integral equation

$$\phi(x) \;=\; \lambda\int K(x, y)\; \phi(y)\, dy \;+\; f(x) \tag{3.23}$$

Where in this case $K$ in degenerate, any solution will be of the form:

$$\phi(x) \;=\; \lambda \sum_{p=1}^{n} a_p(x)\phi_p + f(x)$$

(Substituting, we have)

$$= \; \lambda\int \sum_{p=1}^{n} a_p(x) b_p(y)\phi(y)dy \;+\; f(x)$$

$$= \; \lambda \sum_{p=1}^{n} a_p(x) b_p(y)\left[\lambda \sum_{q=1}^{n}\phi_q a_q(y) + f(y)\right]dy \;+ f(x)$$

$$= \; \lambda \sum_{p=1}^{n} a_p(x)\,\lambda \sum_{q=1}^{n} \int b_p(y)\, a_q\,(y)\phi_q\; dy$$

$$+ \; \sum_{p=1}^{n} a_p(x)\int b_p(y)\; f(y)\, dy + f(x)$$

Set $\quad k_{pq} \;=\; \lambda\int b_p(y)\, a_q(y)\, dy$

and $\quad f_p \;=\; \int b_p(y)\; f(y)\, dy$

Thus, $\quad \phi(x) \;=\; \lambda \sum_{p=1}^{n} a_p(x)\phi_p \;+\; f(x)$

$$= \; \lambda \sum_{p=1}^{n} a_p(x)\left[\lambda \sum_{q=1}^{n} k_{pq}\; \phi_q \;+\; f_p\right] + f(x)$$

$$\therefore \;\; \phi_p \;=\; \lambda \sum_{q=1}^{n} K_{pq}\; \phi_q + f_p \tag{3.24}$$

$$\text{i.e. } \phi_p \;-\; \lambda \sum_{q=1}^{n} K_{pq}\; \phi_q \;=\; f_p \tag{3.25}$$

The above equations is a finite system of linear algebraic equations with matrix $A \;=\; (k_{pq})$.

The solution depends on whether or not det $(\mathrm{I} - \lambda A)$is zero.

Set $\quad \wp(\lambda) \;=\; \det(\mathrm{I} - \lambda\, A)$

Then, $\wp(\lambda)$ is a polynomial of degree $n$. If $\lambda$ is not a roof of $\wp(\lambda)$, then, (3.23) has a unique solution.

If you write $d_{pq}$ for the cofactors you will have:

$$\phi_q = \frac{1}{\wp(\lambda)} \sum_{p=1}^{n} d_{pq} \; f_p \qquad (3.26)$$

The solution of equation (3.23) is then,

$$\phi(x) = f(x) + \lambda[\wp(\lambda)]^{-1} \sum_{p-1}^{n} a_p(x) d_{pq}(\lambda) \int \sum_{q=1}^{n} b_q(y) f(y) dy$$

$$= f(x) - \lambda[\wp(\lambda)]^{-1} \int \wp(x, y; \lambda) f(y) dy \qquad (3.27)$$

Where

$$\wp(x, y; \lambda) = -\sum_{p=1}^{n} \sum_{q=1}^{n} d_{pq} \; a_p(x) \; b_q(y) \qquad (3.28)$$

and $R(x, y; \lambda) = +\dfrac{\wp(x, y; \lambda)}{\wp(\lambda)}$ (3.29)

is the resolvent kernel

i.e. $\phi(x) = f(x) - \lambda \int R(x, y; \lambda) f(y) dy$ (3.30)

**Examples 3.5**

Solve the integral equation:

$$\phi(x) = \lambda \int_o^1 (1 + xt) \phi(t) dt + f(x)$$

Let $\phi(x) = \phi_o + \phi_1 x + f(x)$

$$= \lambda \int_o^1 (1 + xt)[\phi_o + \phi_1 t + f(t)] dt + f(x)$$

$$= \lambda\left(\phi_o + \frac{\phi_1}{2} + f_o\right) + \lambda x\left(\frac{\phi_o}{2} + \frac{\phi_1}{3} + f_1\right) + f(x)$$

Where $f_r = \int_o^1 t^r f(t) \; dt$

Equating powers of $x$ and solving for $\phi_o$ and $\phi_1$ if follows that:

$$\phi_o\left(\lambda^2 - 16\lambda + 12\right) = \left[-4\lambda(\lambda - 3)\, f_o + 6\lambda^2\ f_1\right]$$

$$\phi_1\left(\lambda^2 - 16\lambda + 12\right) = \left[6\lambda^2\ f_o - 12\lambda(\lambda - 1)\, f_1\right]$$

The Eigenvalue, are given by the roots of the equation

$$\lambda^2 - 16\lambda + 12 = 0$$

If $\lambda$ is one of the Eigenvalue, say $8 + \sqrt{52},$ a solution is possible only if

$$O = \int_o^1 f(x)\left[\frac{8+\sqrt{52}}{2} - \left(7+\sqrt{52}\right)x\right]dx$$

and the solution is indefinite by an arbitrary multiple of $4 + \sqrt{13} - \left(7 + \sqrt{52}\right)x$

## 3.2 Symmetric Kernels

$K(x, y) = k(y, x)$ and $K$ is continuous

**Theorem 3.2**

Let $K(x, y)$ be symmetric and continuous (and not identically zero). Then, $K$ has at least one Eigenvalue.

**Proof**

We note that the iterated kernels $k_u(x, y)$ are also symmetric and not identically zero.

Suppose result is not true.

Let us assume that $R(x, y; \lambda)$, the resolvent kernel is an integral function and the series is also convergent for all, $\lambda$. may be integrated term by term.

Now set, $U_n = \int K_n\,(x, x)\,dx$

Then, $U_2\lambda^2 + U_4\lambda^4 + .......$ is absolutely convergent.

Now, we have $U_{n+m} = \iint K_n\,(x, z)\ K_m\,(z, x)\,dxdx$

and $U_{2n} = \iint K_n^2\,(x, z)\,dxdz$ (3.40)

Now, $\iint \left[\alpha\ K_{n+1}\,(x, z) - \beta\ K_{n-1}\,(x, z)\right]^2\ dxdz \geq o$

i.e.

$$\alpha^2\, U_{2n+2} - 2\alpha\beta\, U_{2n} + \beta^2 U_{2n-2} \geq o \qquad (3.41)$$

for all real $\alpha,\ \beta$

$$\therefore\ \ U_{2n}^2 \ \leq\ U_{2n+2}\, U_{2n-2} \qquad (3.4.2)$$

Form equation (3.40) none of $U_{2n}$ is zero as $K_n$ is not identically zero.

$$\therefore\ \ \frac{U_{2n+2}}{U_{2n}} \ \geq\ \frac{U_{2n}}{U_{2n-2}}$$

Now, consider series $\sum \lambda^{2n} U_{2n}$ assumed convergent.

The ratio of term is

$$\frac{U_{2n+2}\lambda^{2n+2}}{U_{2n}\kappa^2} \ =\ \frac{U_{2n+2}}{U_{2n}}\,\lambda^2$$

This ration is $\geq\ \dfrac{U_4}{U_2}\,\lambda^2$ from (3.43)

Thus, for $\dfrac{U_4}{U_2}\,\lambda^2 \geq 1,$ the forms in the (3.44) series are non-increasing, so as it is a series of positive terms, the series is divergent. This is a contradiction.

Thus, we have seen that poles of $R(x,\ y;\ \lambda)$ correspond to Eigenvalue and so K has at least one Eigenvalue.

From the equation (3.44) $\dfrac{U_4}{U_2}\,\lambda^2 \geq 1,$ the smallest Eigenvalue $\lambda_1$ is such that

$$\lambda_1 \ \leq\ \sqrt{\frac{U_4}{U_2}} \quad (3.45)$$

**Theorem 3.3**

If $k(x,\ y)$ is symmetric and continuous, the:

- number of Eigenfunctions corresponding to each Eigenvalue is finite
- Eigenfunction corresponding to different Eigenvalue are orthogonal
- Eigenvalue is real.

## 4.0 CONCLUSION

Eigenvalues and Eigenfunctions can be found for integral equations of the form $\phi(x) = \lambda\int\sum_{p=1}^{n} a_p(x)\,b_p(y)\,d(y)dy$.

## 5.0 SUMMARY

Many homogeneous equations can be solved by determination of their Eigenvalue.

## 6.0 TUTOR-MARKED ASSIGNMENTS

1. Under what Sturm-Liouville problem assumptions are Eigenfunction corresponding to different Eigenvalue orthogonal?

**2.** Solve

$$\phi(x) = \lambda\int_0^1 (1+xt)\phi(t)dt + mf(x); 0 \le x \le 1$$

## 7.0 REFERENCES/FURTHER READING

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# UNIT 3 REPRESENTATION OF A FUNCTION BY A SERIES OF ORTHOGONAL FUNCTIONS

**CONTENTS**



## 1.0 INTRODUCTION

In this unit, we shall take a look at orthogonality of systems and show that Fourier coefficients exist for continuous orthogonal systems and that orthogonal system can be represented by a series of orthogonal functions.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the Hilbert – Schmidt theorem;
- state the Convergence theorem;
- prove that functions can be represented by series of orthogonal functions;
- expand $K$ in a series of Eigen functions; and
- define positive kernels.

## 3.0 MAIN CONTENT

## 3.1 Representation of a Function by a Series of Orthogonal Functions

### 3.1.1 Lemma 3.4

Let $\{\phi_n\}$ be an orthogonal system, and let $f$ be continuous.

Set $\alpha_n = \int_I f(x)\phi_n(x)\,dx$ (3.46)

Then, $\sum \alpha_n^2 \le \int_I f^2(x)\,dx$ (3.4.7) and $\alpha_n^{1s}$ are known as the Fourier's coefficient.

**Proof:**

Take any $N$. consider

$$\int_I \left[ f(x) - \sum_1^N \alpha_n \phi_n(x) \right]^2 dx \geq O \tag{3.48}$$

i.e. $\int \left[ f^2(x) - 2 \sum_1^n \alpha_n \int f(x)\phi_n(x) + \sum_1^n \alpha_n^2 \right] dx \geq o$

i.e. $\int_I f^2(x)dx \geq \sum_1^N \alpha_n^2$ (3.49)

On noting that N is arbitrary

$$f(x) = \sum_1^\infty \alpha_n \phi_n(x) \tag{3.50}$$

We now consider that coefficients, $\alpha_n$, give the best fir in the sense that

$$\int [f(x) - \sum C_n \phi_n(x)]^2 dx \tag{3.51}$$

is a minimum.

The answer is that $C_n = \alpha_n$, the Fourier coefficients.

To see this, set

$$I_n^* = \int [f(x) - \sum C_n \phi_n]^2 dx \tag{3.52}$$

$$I_n = \int [f(x) - \sum \alpha_n \phi_n]^2 dx \tag{3.53}$$

Then, we show that $I_n^* \geq I_n$. For we have,

$$I_n^* = \int f^2(x)\,dx - 2\sum C_n \int f\,\phi_n\,dx + \sum C_n^2$$

$$\int f^2(x)\,dx - 2\sum \alpha_n C_n + \sum C_n^2$$

$$I_n = \int f^2(x)\,dx - 2\sum \alpha_n^2 + \sum \alpha_n^2$$

$$\therefore\ I_n^* - I_n = \sum [(C_n^2 - \alpha_n^2) - 2\alpha_n (C_n - \alpha_n)]$$

$$= \sum (C_n - \alpha_n)^2 \geq o \tag{3.54}$$

As asserted.

**Definitions 3.5**

The set of orthogonal system $\{\phi_n\}$ is said to be complete if

$$\lim_{n\to\infty}\int_I\left[f(x)-\sum\alpha_n\ \phi_n\right]^2\ dx\ =\ o \qquad (3.55)$$

for every continuous function $f(x)$. In this case, the Bessel's inequality becomes an equality

i.e. $\int f^2(x)\,dx \;=\; \sum\alpha_n^2$

If $\{\phi_n\}$ is complete, we can then roughly represent any function as a sum

$$f(x) \;=\; \sum\alpha_n\ \phi_n$$

the convergence being in the sense of (3.55)

## 3.2 Expansion of $K$ in Eigenfunctions

We examine the possibility of expanding $K$ as a series of Eigenfunctions.

Consider

$$K(x, y) \;=\; \sum\alpha_n\ \phi_n(y)$$

where $\int K(x, y)\phi_n(y)dy \;=\; \alpha_n$

i.e. $\alpha_n \;=\; \lambda_n^{-1}\ \phi_n(x)$

Then, we will have

$$K(x, y) \;=\; \sum\frac{\phi_n(x)\phi_n(y)}{\lambda_n} \qquad (3.56)$$

This is valid independent of completeness of $\{\phi_n\}$

### 3.2.1 Definitions 3.6 (Positive Kernels)

A kernel is said to be positive if

$$T(\phi, \phi) = \iint K(x, y)\ \phi(x)\phi(y)\,dxdy > o \quad (3.57)$$

for all $\phi$ such that $\int \phi^2(x)\,dx \neq o$ (3.58)

It is easily to see that the Eigenvalues are strictly positive.

### 3.2.2 Theorem 3.7: Convergence

If $K(x, y)$ is positive, then

$$K(x, y) = \sum_{1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}$$

the series being absolutely and uniformly convergent.

### 3.2.3 Theorem 3.8: Hilbert – Schmidt Theorem

Suppose $f(x)$ can be written in the form

$$f(x) = \int K(x, y)\phi(y)\,dy \quad (3.59)$$

where $K$ is symmetric and continuous,

Then $f = \sum \alpha_n\ \phi_n$ where the series is absolutely and uniformly convergent and

$$\alpha_p = \int_a^b f(x)\phi_p(x)\,dx \quad (3.60)$$

A convergence of the above theorem is another formula for the Resolvent $R$.

Consider the equation:

$$\phi(x) - \lambda \int K(x, y)\ \phi(y)\,dy = f(x) \quad (3.61)$$

Then,

$$\phi - f = \lambda \int k(x, y)\phi(y)\,dy$$

Thus, $\phi - f$ satisfies the condition of theorem 3.8 and we can write

$$\phi(x) - f(x) = \sum \alpha_n\ \phi_n$$

where $\alpha_n = \int [\phi(x) - f(x)]\, \phi_n(x)\, dx$

$= \beta_n - \gamma_n = \int f(x)\, \phi_n(x)\, dx$

Multiply 3.61 by $\phi_n(x)$ and integrate and change order of integration. This given

$$\int \phi(x)\, \phi_n(x)\, dx - \lambda \int \phi(x) dx \int k(x, y)\, \phi_n(y)\, dy$$

$$= \int f(x)\, \phi_n(x)\, dx \tag{3.62}$$

Thus, $\beta_n - \dfrac{\lambda\, \beta_n}{\lambda_n} = \gamma_n$

i.e. $\beta_n = \dfrac{\lambda_n}{\lambda_n - \lambda}\, \gamma_n$, Hence, $\alpha_n = \dfrac{\lambda}{\lambda_n - \lambda}\, \gamma_n$

$\therefore \ \phi(x) - f(x) = \lambda \sum \dfrac{\gamma_n}{\lambda_n - \lambda}\, \phi_n(x)$

$$\text{i.e.} \quad \phi(x) = f(x) + \lambda \sum \frac{\lambda_n}{\lambda_n - \lambda}\, \phi_n(x) \tag{3.63}$$

This gives the solution of 3.61 in terms of the Eigenfunctions of $\phi_n(x)$. From 3.63, we have

$$\phi(x) = f(x) - \lambda \sum \frac{\phi_n}{\lambda - \lambda_n} \int \phi_n(y)\, f(y)\, dy$$

$$= f(x) - \lambda \int \left[ \sum \frac{\phi_n(x)\, \phi_n(y)}{\lambda - \lambda_n} \right] f(y)\, dy$$

$$= f(x) - \lambda \int R(x, y; \pi)\, f(y)\, dy$$

$$\text{Where } R(x, y; \lambda) = \sum \frac{\phi_n(x)\, \phi_n(y)}{\lambda - \lambda_n} \tag{3.64}$$

## 4.0 CONCLUSION

Fourier's coefficients exist for orthogonal systems which are continuous.

## 5.0 SUMMARY

The Hilbert-Schmidt theorem states that when a kernel is positive, a series can be derived which is absolutely and uniformly convergent, and kernel can be represented by a series of Eigenfunctions.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Derive an expression for the Fourier coefficient of the continuous orthogonal system $\{\phi_n\}$?

2. Show that the orthogonal system $\{\phi_n\}$ is complete if
   $\lim_{n\to\infty} \int_I \left[ f(x) - \sum \alpha_n \, \phi_n \right]^2 dx = o$?
   that is, Bessel's inequality becomes an equality.

3. If $T(\phi, \phi) = \iint K(x, y) \, \phi(x) \phi(y) \, dxdy > o$, can we deduce if the associated kernel is positive or negative? Prove it?

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