MODULE 4

UNIT 1 CALCULATION OF 1ST EIGENVALUE

CONTENTS

1.0 INTRODUCTION

Transforms are used to solve equations for which transforms exist while the inverse transform is a convolution. Suitable conditions exist for the transform of a convolution to become the point-wise product of transforms which means that convolution in one domain is the point-wise multiplication in another domain.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply the convolution theorem;
- calculate the first eigenvalue of an integral equation;
- use the variational formula;
- recognise Integral Laplace transforms as transforms; and
- derive the solution of integral equations using inverse Laplace transform.

3.0 MAIN CONTENT

3.1 Calculation of 1st Eigenvalue

The modes of vibration in systems are often of great importance. A powerful and simple method for finding them is provided by variational formula.

Let $\phi_1, \phi_2, \ldots$ be Eigen functions and $|\lambda_1| < |\lambda_2| < \ldots)$ be the corresponding Eigenvalue.

Set

$$J\,(\phi,\phi) \;=\; \iint K(x,y)\,\phi(x)\,\phi(y)\,dxdy$$

Suppose now that ϕ is arbitrary. Then, by the linear formula

$$K(x,y) \;=\; \frac{\sum \phi_n(x)\,\phi_n(y)}{\lambda_n}$$

We have:

$$J(\phi,\phi) \;=\; \iint \frac{\sum \phi_n(x)\,\phi_n(y)}{\lambda_n}\,\phi(x)\,\phi(y)\,dxdy$$

$$= \;\sum \frac{\beta_n^2}{\lambda_n} \;\le\; \sum \left|\frac{\beta_n^2}{\lambda_n}\right|$$

Then, $J\,(\phi,\phi) \;\le\; \sum \frac{\beta_n^2}{|\lambda_n|} \;\le\; \frac{1}{|\lambda_1|}\sum \beta_n^2$

$$\le\; \frac{1}{|\lambda_1|}\int \phi^2(x)dx \quad \text{(Bessel equation)}$$

$$\therefore\; |\lambda_1| \;\le \frac{\int {}^2(x)dx}{J\,(\phi,\phi)} \tag{3.65}$$

where λ_1 is the smallest Eigenvalue and ϕ is arbitrary. Similar results may be obtained for the higher Eigenvalues. However, the first is usually, the most important. ϕ is chosen to make $J\,(\phi,\;\phi)$ a maximum and a normal function.

This given an estimate of a bound for λ_1 which usually is fairly accurate.

Example 3.7

Consider the kernel T in the square

$o \le x \le 1,\quad o \le y \le 1$ where

$$T(x,y) \;=\; \begin{cases}(1-x)y & o \le y \le x \le 1\\ (1-y)x & o \le x \le y \le 1\end{cases}$$

By differentiating the equation

$$\phi(x) \;-\; \lambda\int_o^1 \mathrm{T}(x,y)\,\phi(y)\,dy \;=\; 0$$

If is easy to see that if reduces to

$$\phi'' + \lambda \phi(x) = 0, \quad \phi(o) = \phi(1) = o$$

The Eigen function are $\sqrt{2}\,\sin n\pi x$ (normalized and Eigenvalue are $\lambda_n = (n\pi)^2$

The linear formula given

$$\mathrm{T}(x, y) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi y}{n^2\pi^2}$$

We shall now consider the application of 3.65 to the determination of 1st Eigenvalue

$$\left(\lambda_1 = \pi^2 = 9.869\right)$$

First guess $\phi = 1$

$$J(\phi, \phi) = \int_o^1 \left[(1-x)\int_o^x y\,dy + x\int_x^1 (1-y)\,dy\right] dx$$

$$= \frac{1}{12}$$

We get $\lambda_1 = 1\big/\frac{1}{12} = 12$

Second guess $(R_i\ t_3)$

Choose ϕ to be a step function $\phi = 0$ except for $o < x < 1-\alpha$ where $\phi = \beta$.

Choose ϕ normalised. Then, $\beta = (1-2\alpha)^{-\frac{1}{2}}$ one find that

$$J(\phi, \phi) = \frac{1}{12}\left(1 + 2\alpha - 8\alpha^2\right)$$

This has a maximum at $\alpha = \frac{1}{8}$, when

$$J(\phi, \phi) = 3/32$$

Then, $\lambda_1 \leq \dfrac{1}{3/32} = \dfrac{32}{3} = 10.67$

The estimate is considerably improved, and the choice of a nose complicated ϕ will lead to a nose accurate estimate.

3.2 Integral Transforms; Laplace Transforms

If $f(t)$ is throughout piecewise, continuous, bounded variation and of exponential order, i.e. $\exists$ M_o, so, э

$f(t) \leq M_o e^{sot}$

and if we define

$$F(s) = \int_o^\infty e^{-st} f(t)\, dt \qquad (4.1)$$

S may be complex, then, $F(s)$ is known as the Laplace transform of f and is defined when the integral is absolutely convergent for some so, then, it is also for S such that Res > Redo

The largest half-plane in which the integral is absolutely convergent is called the half-plane of convergence. The following hold in this half-plane:

i. $£\{af + bg\} = a£\{f\} + b£\{\text{g}\}$ (4.2)

ii. $£\{f^2(t)\} = S^n F(s) - S^{n-1} f(o^+)$.
$\ldots - F^{n-1} \quad (o^+)$ (4.3)

iii. $£\{\text{e}^{\text{at}} f(t)\} = F(s-a)$ (4.4)

iv. $£\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$ (4.5)

$F(o^t)$ denotes limit from right

3.3 Convolution Theorem

We define a new function $h(t)$ by

$$h(t) = \int_o^t g(u) f(t-u)\, du = f * g \qquad (4.6)$$

$h(t)$ is called the convolution product of f and g and is written $f * g$ so that we have

$$\int_o^\infty e^{-st} h(t)\, dt = \int_o^\infty e^{-st} (f * g)\, dt \qquad (4.7)$$

$$= \int_o^\infty e^{-st} f(t)\, dt \int_o^\infty e^{-su} g(u) du = F(s) C_1(s)$$

3.4 Inverse Laplace Transform

$$f(t) = £^{-1}\{F(s)\} = \frac{1}{2\pi_i}\int_{c-i\infty}^{c+i\infty} F(s)\,ds \quad (4.8)$$

$$= \sum\{\text{residues of } F(x)e^{st} \text{ at poles of } F(s)\}$$

Where C is some real number which is greater than the real part of all the poles of $F(s)$. We can however use any other alternative method to obtain $f(t)$.

4.0 CONCLUSION

Kernel can be solved by applying Laplace transform if the transform exists.

5.0 SUMMARY

Laplace transform is defined only when an integral is absolutely convergent and the largest half-plane in which the integral is absolutely convergent is called the half-plane of convergence.

6.0 TUTOR- MARKED ASSIGNMENT

1. Write an expression for the kernel T in the square defined below and find its first Eigenvalue?

   $-1 \le x \le 2, \qquad 0 \le y \le 3$

2. Do you recognise the transform below? Which transform is it?

   $F(s) = \int_o^\infty e^{-st} f(t)\,dt$

3. What is the relationship between an inverse Laplace transform and a convolution?

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UNIT 2 THE APPLICATION OF THE TRANSFORM

CONTENTS

1.0 INTRODUCTION

Laplace and Fourier integral transforms are used to solve integral equations for which the transform exists and this is demonstrated in this unit via worked examples.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply Laplace transform through worked examples; and
- solve integral equations by the method of Fourier integral transforms.

3.0 MAIN CONTENT

3.1 The Application of the Transform

Example 4.1

Solve the equation:

$$\phi^{11} + 5\ \phi^{1} + 6\phi = e^{-t} \quad t \geq o$$

$$\phi(o) = 2, \quad \phi^{1}(o) = 1$$

Now, let $£(\phi) = \bar{\phi}(s)$ so that

$$£\{\phi^{11}\} = s^2\bar{\phi} - s\,\phi(o) - \phi^1(o) = s^2\bar{\phi}(s) - 2s - 1$$

and $£\{e^{-t}\} = \dfrac{1}{s+1}$

$$\therefore \quad s^2\,\bar{\phi} - 2s - 1 + 5(s\bar{\phi} - 2) + 6\bar{\phi} = \frac{1}{s+1}$$

i.e. $\left(s^2 + 5s + 6\right)\bar{\phi} \;=\; 25 \;+\; 11 \;+\; \dfrac{1}{s+1}$

i.e. $\bar{\phi}(s) \;=\; \dfrac{2s^2 + 13s + 12}{(s+1)(s+2)(s+3)}$

Hence, the poles are not $-1, -2, -3$

The residue at $s = -1$ is $\operatorname{Re} s_{s=\alpha} = \bar{\phi}^{(s)(s-\alpha)} 1_{s=\alpha}$

$$\frac{2-113+12}{1\times 2} e^{-t} \quad = \quad \frac{1}{2} e^{-t}$$

That is -2 is $6\, e^{-2t}$

and -3 is $-\dfrac{9}{2} e^{-3t}$

Thus, $\phi(t) \;=\; \dfrac{1}{2}\left\{e^{-t} + 12e^{-2t} - 9e^{-3t}\right\}$

Example 4.2

Consider the Volterra equation:

$$f(x) - \int_o^x k(x-y)\, f(y) dy \quad = \quad g(x)$$

We want to use Laplace transform to get a solution.

The equation can be written in the form of

$$f - k * f \;=\; g$$

Take the Laplace transform of both sides to give

$$\bar{f} \;-\; \bar{k}\; \bar{f} \;=\; \bar{g}$$

i.e. $\bar{f}\left(1-\bar{k}\right) = \bar{g} \qquad \therefore \quad \bar{f} \;=\; \bar{g}\big/ 1-\bar{k}$

Thus,

$$f \;= £^{-1}\left\{\frac{\bar{g}}{1-\bar{k}}\right\} \quad = \quad \frac{1}{2\pi_i}\int_{\Gamma} \frac{\bar{g}}{1-\bar{k}}\, e^{st}\, ds$$

Take $k(t) \;=\; \lambda e^{t}$ for example, then

$$\bar{k}(s) \;=\; \frac{\lambda}{s-1}$$

But $\dfrac{1}{1-\bar{k}} \;=\; 1+\dfrac{\bar{k}}{1-\bar{k}}$

$$f \;=\; £^{-1}\left\{\frac{\bar{g}}{1-\bar{k}}\right\} \;=\; £^{-1}\left\{\bar{g}+\frac{\bar{k}\,\bar{g}}{1-\bar{k}}\right\}$$

$$= \; g \;+\; £^{-1}\left\{\frac{\lambda\,\bar{g}}{s-1-\lambda}\right\} \;=\; g + £^{-1}\left\{\bar{h}\;\bar{g}\right\}$$

Where $\bar{h} \;=\; \dfrac{\lambda}{s-1-\lambda}$ and $h \;=\; \lambda\, e^{(1+\lambda)t}$

Hence,

$$f(x) \;=\; g(x)+\lambda\int_{o}^{x} e^{(1+\lambda)(t+u)}\; g(u)\,du\,.$$

Example 4.3

Solve the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} \;-\; \frac{\partial^2\phi}{\partial t^2} \;=\; o \qquad (o \le x \le l, \quad t \ge o)$$

$$\phi(x, o) \;=\; o \qquad o \le x \le \ell$$

$$\frac{\partial\phi}{\partial t}(x, o) \;=\; o \qquad o \le x \le l$$

$$\phi(o, t) \;=\; o \qquad t \ge o$$

$$\frac{\partial\phi}{\partial x}(l, t) \;=\; a \qquad t \ge o$$

Here we want a solution for $t \ge o$ and for a finite range of x. Take Laplace transform w.r.e t (since the t – interval is semi-infinite)

Write $\bar{\phi}(x, s) = \int_{0}^{\infty} e^{-st}\phi(x, t)\,dt$

Take the Laplace transform to give

$$\int_0^{\infty} e^{-st}\,\frac{\partial^2\phi}{\partial x^2}(x,\,t)dt - \int_0^{\infty} e^{-st}\,\frac{\partial^2\phi}{\partial t^2}(x,\,t)dt \;=0$$

But $\int_0^{\infty} e^{-st}\,\frac{\partial^2\phi}{\partial x^2}(x,t)dt \quad = \quad \frac{\partial^2}{\partial x^2}\int_0^{\infty} e^{-st}\phi(x,t)dt$

$$= \quad \frac{\partial^2\overline{\phi}}{\partial x^2}(x,\,s)$$

and $\int_0^{\infty} e^{-st}\,\frac{\partial^2\phi}{\partial x^2}(x,t)dt \quad = \quad s^2\;\overline{\phi}(x,s) - S\phi(x,\,0) + \frac{\partial\phi}{\partial t}(x,0)$

$$= \quad S^2\overline{\phi}(x,\,s)$$

from the boundary condition

Hence, we get

$$\frac{\partial^2\overline{\phi}}{\partial x^2}(x,\,s) \;-\; s^2\,\overline{\phi}(x,\,s) = 0$$

This is now an ordinary differential equation for $\overline{\phi}$ and to solve it, we need two boundary conditions, we have

$$\overline{\phi}\,(0,\,s) \;=\; 0$$

and $\frac{\partial\overline{\phi}}{\partial x}(\ell,\,s) \;=\; \int_0^{\infty} e^{-st}\,\frac{\partial\phi}{\partial x}(\ell,\,t)dt$

$$= \quad \int_0^{\infty} ae^{-st}dt = \frac{a}{s}$$

We this, solve the following system

$$\frac{\partial^2\overline{\phi}}{\partial x^2}(x,\,s) - s^2\;\overline{\phi}\,(x,\,s) \;=0$$

$$\overline{\phi}(0,\,s) = 0 \qquad \text{and} \quad \frac{d\overline{\phi}(\ell,\,s)}{dx} = \frac{a}{s}$$

The solution is

$$\overline{\phi} = A(s)\sinh sx \;+\; B(s)\cosh sx$$

From the first boundary condition $B(s) = 0$ and from the second condition, we have

$$A(s)\, s \cosh sl \;=\; \frac{a}{s}$$

$$A(s) \;=\; \frac{a}{s^2 \cosh sl}$$

Here, $\bar{\phi}(x,s) \;=\; \dfrac{a \sinh sx}{s^2 \cosh sl}$

and $\phi(x,t) \;=\; \dfrac{a}{2\pi i} \displaystyle\int_{r^1} \frac{\sinh sx}{s^2 \cosh sl}\, e^{st}\, ds$

where τ lies to the sight of the poles. The integral has poles at $s = 0$ and at the zeros of $\cosh sl$. Consider first $s = 0$

$$\bar{\phi}(x,s)\, e^{st} = \frac{1}{s^2}\left[sx + \frac{(sx)^3}{6} + \cdots \right]\left[1 - \frac{(sl)^2}{2} \cdots \right][1 + st + \cdots]$$

$$= \frac{1}{s^2}\left[sx + 0\left(s^2\right)\right]$$

Simple pole at $s = 0$ with reside x.

Now, consider points whose $\cosh ls \;= 0$

$$S = \frac{\pm i\,(2n+1)\pi}{2l}$$

The poles are simple once. We may use the formula:

$$\operatorname*{Re}_{s=a} s \;= \frac{f(a)}{g^1(a)}$$

Thus,

$$\operatorname*{Re}_{s=\frac{(2n+1)i\pi}{2l}} s \;=\; \frac{Sinh\left[\dfrac{(2n+1)i\pi x}{2l}\right] e^{i\frac{(2n+1)\pi t}{2l}}}{\ell\left[\dfrac{(2n+i)i\pi}{2l}\right]^2 Sinh\left[\dfrac{(2n+1)\pi i}{2}\right]}$$

$$= \; -\frac{4\ell(-1)^n}{\pi^2}\;\frac{Sinh\left[\dfrac{(2n+1)\pi ix}{2l}\right] e^{i\frac{(2n+1)\pi t}{2l}}}{(2n+1)^2\, i}$$

Evidently, poles are complex conjugates, so we require twice the seal part. Hence,

$$\phi(x, t) = ax - \frac{8al}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\, Sin\left[\frac{(2n+1)\pi x}{2l}\right]$$

The Laplace transform is suitable for problems with a semi-infinite domain for the independent variable. It is also necessary that the (differential) equation should have constant coefficients.

3.2 Fourier Integral Equations

If $f(x)$ is a continuous function, then,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} e^{iwx} F(w)\,dw \quad (4.10)$$

where $$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwu} f(u)\,du \qquad (4.11)$$

Equation 4.11 gives the solution of the integral 4.10 for F and vice versa. If $f(x)$ is seal, using the odd property of sine, $\sin wx$ and the even property of $\cos wx$ we have, if

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\cos wx\,\phi(w)\,dw, \quad 0 \le x \qquad (4.12)$$

Then,

$$\phi(w) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\cos wx\; f(x)\,dx, \quad 0 \le w \qquad (4.13)$$

$\phi(w)$ and $f(x)$ are the $\cos ine$ transforms of one another. If

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\sin wx\,\phi(w)\,dw, \quad 0 \le x \qquad (4.14)$$

Then,

$$\phi(w) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\sin wx\; f(x)\,dx, \quad 0 \le w \qquad (4.15)$$

$\phi(w)$ and $f(x)$ are the sine transforms of one another.

Example 4.1

Solve the integral equation

$$\frac{a}{a^2 + x^2} = \int_0^\infty \cos wx\, \phi(w)dw, \quad a > 0$$

$$\phi(w) = \left(\frac{2}{\pi}\right)\int_0^\infty \frac{a\cos wx}{a^2 + x^2}dx = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{2ia\, e^{iwx}}{a^2 + x^2}\, dx$$

because $\sin wx$ is odd in x

Evaluation of the integral by the methods of the complex integral calculus given

$$\phi(w) = e^{-wa}, \quad w > 0$$

Example 4.2

Solve the integral equation

$$\phi(x) = \lambda\int_0^\infty \cos wx \;\; \phi(w)dw$$

$\phi(x)$ is an even function of x

Because the inverse of a cosine transform is another $\cos ine$ transformation, we look for a solution of the form

$$\phi(x) = U(x) \pm V(x)$$

where $V(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos wx\, U(w)dw$

Thus,

$$\phi(x) = U(x) \pm \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos wx\, U(w)dw$$

$$= \lambda\int_0^\infty \cos wx \left[U(w) \pm \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos wt\, U(t)dt\right]dw$$

$$= \lambda\int_0^\infty \cos wx\, U(w)dw \pm \left(\frac{\pi}{2}\right)^{1/2} \lambda\, U(x)$$

This is true if $\lambda = \pm\left(\frac{2}{\pi}\right)^{1/2}$ Thus, to $\lambda = \left(\frac{2}{\pi}\right)^{1/2}$, there corresponds a solution $U(x)+V(x)$ and to $\lambda = -\left(\frac{\pi}{2}\right)^{1/2}$, there corresponds a solution $U(x)-V(x)$.

This solution will be valid, provided all the integrals exist; U is arbitrary. In this case, the two Eigenvalues $\lambda = \pm\left(\frac{2}{\pi}\right)^{1/2}$, there exist an infinite of Eigenfunctions.

Example 4.3

Solve the integral equation

$$\phi(x) = f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, \phi(y)dy$$

If $\lambda = \pm 1$, there will not in generally be any solution.

This follows example 4.2

Take the transform of the equation to give

$$\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, \phi(y)dy = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)dy + \lambda\, \phi(x)$$

It follows that

$$\phi(x) = f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)dy + \lambda^2\phi(x)$$

i.e.

$$\left(1-\lambda^2\right)\phi(x) = f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)\, dy$$

and this solution is valid provided that the integral converge. Now, if $1-\lambda^2 = 0$ and $f(x)$ is a function such that

$$f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)dy = 0$$

It follows that $\phi(x)$ can be any function for which the integral converge.

6.0 TUTOR- MARKED ASSIGNMENT

(1) Solve the integral equation.

$$\frac{x}{x^2 + a^2} = \int_0^\infty \sin wx\phi(w)dw \qquad a > 0$$

(2) Find the Eigenvalues and Eigen functions of the integral equation.

$$\phi(x) = \lambda\int_0^\infty \sin xy\ \phi(y)dy$$

(3) Find the solution of the integral equation.

$$\phi(x) = e^{-ax} + \lambda\int_0^\infty \sin xy\ \phi(y)dy, \qquad a > 0$$

$$\pi\lambda^2 \neq 2$$

(4) Find the integral equation:

$$\frac{P}{P^2 + a^2} = \int_0^\infty e^{-pt}\ f(t)dt \qquad a > 0$$

(5) Solve the integral equation:

$$\phi(x) = f(x) + \lambda\int_{-\infty}^\infty K(x-y)\phi(y)dy$$

(6) Solve the integral equation:

$$\frac{1}{(x+a)^2} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{g(y)dy}{x-y}$$

4.0 CONCLUSION

Transforms are a useful mathematical tools for solving integral equations for which the applicable transforms exist.

5.0 SUMMARY

A Laplace transformation is applicable for problems with a semi-infinite domain for the independent variable.

6.0 TUTOR-MARKED ASSIGNMENT

1. Solve $\dfrac{b^2+x^2}{b}=\int_0^\infty \sin wx\phi(wx)dx, 0 \lhd b \lhd \pi$

2. Solve $\phi(t)=\lambda\int_0^\infty \sin wt\ \phi(f)df$

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